Recent Work in Epistemic Logic

Logic Group Colloquium

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Philosophy
UC Berkeley

March 1, 2013

- Preliminaries
- Basic Epistemic Logic
  - Modeling Knowledge
  - Common Knowledge
  - Information Update
- Mathematical Themes
  - Preservation Theorems
  - Uniform Substitution
  - Relations to FOL
- Conclusion

What is Epistemic Logic?

Epistemic logic provides a formal framework for modeling the knowledge of agents. Used by philosophers, theoretical computer scientists, AI researchers, game theorists, and others, epistemic logic has become one of the main application areas for modal logic.

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- **90s-now** *Dynamic* epistemic logic is developed in CS (Jan Plaza) and logic, especially by the "Amsterdam School" around Johan van Benthem.

Now I am interested in bringing epistemic logic back into contact with philosophy, using it as a tool to formalize existing theories of knowledge and develop new and improved ones. See, for example:

Wesley Holliday. Forthcoming.
"Epistemic Logic and Epistemology." *Handbook of Formal Philosophy*.

Wesley Holliday. Forthcoming.
"Epistemic Closure and Epistemic Logic I." *Journal of Philosophical Logic*.

Wesley Holliday. 2012.
"Epistemic Logic, Relevant Alternatives, and the Dynamics of Context."
New Directions in Logic, Language, and Computation, LNCS.

Wesley Holliday and John Perry. Forthcoming.
"Roles, Rigidity, and Quantificaiton in Epistemic Logic." *Trends in Logic*.

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I am working to show how this method gives simple completeness proofs for standard systems of modal logic—good for students.

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Today I'll deal with propositional epistemic logic, but first-order epistemic logic is also philosophically and technically rich.

Definition (Propositional Epistemic Language)

Given sets $\mathrm{At} = \{p, q, r, \dots\}$ and $\mathrm{Agt} = \{a, b, c, \dots\}$ with $|\mathrm{Agt}| = \kappa$, the epistemic language $\mathcal{L}_{\mathsf{EL}}^{\kappa}$ is generated by

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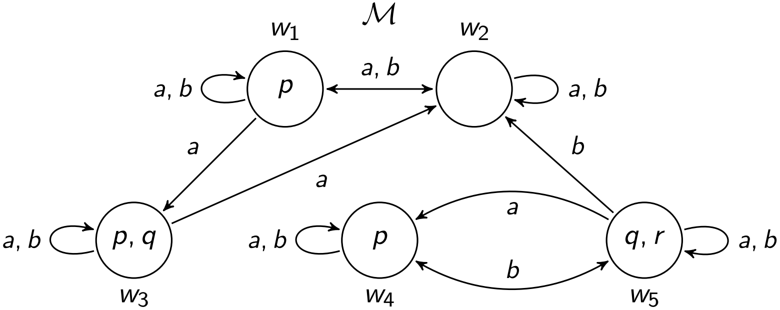
Where $\Box$ is K_a or E_A, $\Box^1\varphi := \Box\varphi$ and $\Box^{n+1}\varphi := \Box\Box^n\varphi$ $(n \geq 1)$.



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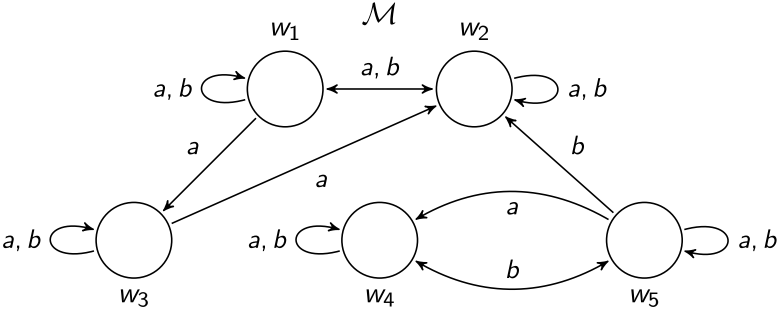

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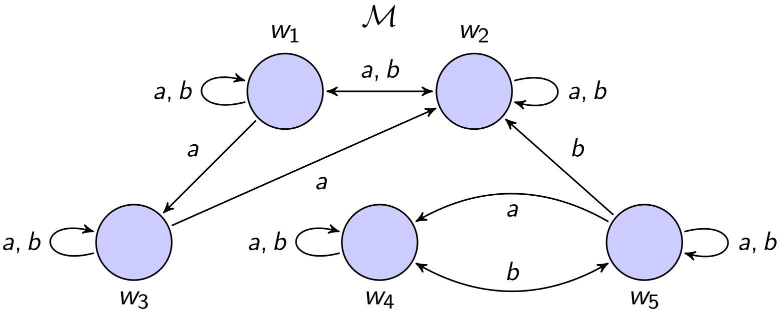

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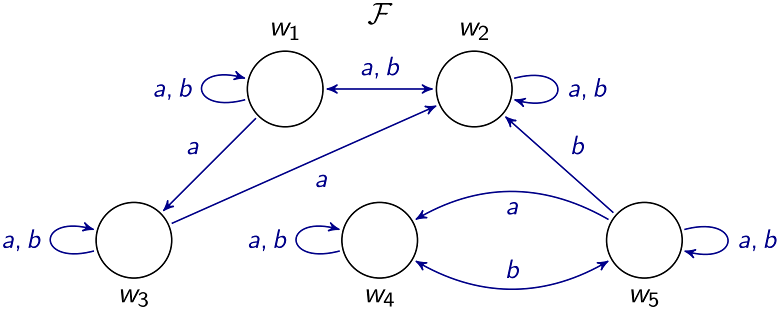
- W is a non-empty set (of "possibilities," "scenarios," "states," or "worlds");



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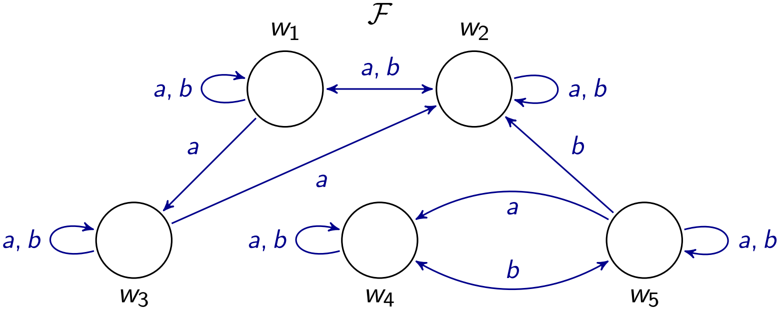
- for each $a \in \text{Agt}$ ($|\text{Agt}| = \kappa$), R_a is a binary relation on W; R_a can be thought of as representing agent a's **uncertainty**.



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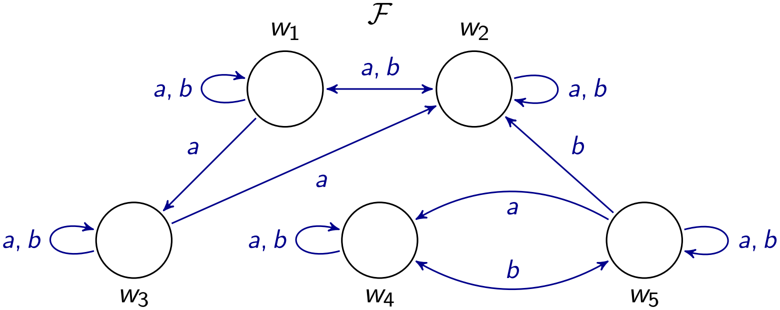

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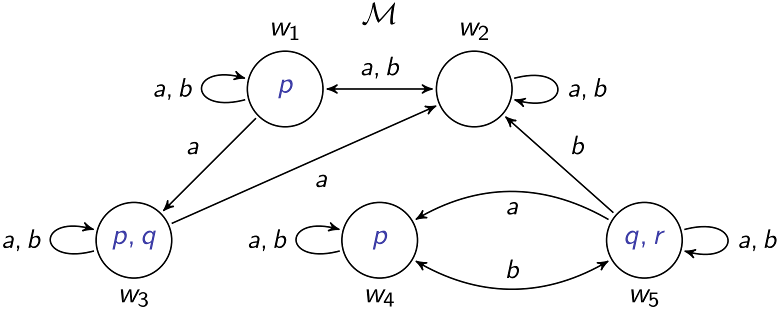

Definition (Epistemic Model)

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- $V : \mathsf{At} \to \wp(w)$ assigns to each atomic sentence the set of scenarios in which it holds. We say $\mathcal{M}$ is *based on* $\mathcal{F}$.

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2. It is *common knowledge* that φ when everyone knows that φ, everyone knows that everyone knows that φ, everyone knows that everyone knows that everyone knows that φ, etc. (inspired by D. Lewis, *Convention*, 1969).

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3. *Knowledge acquisition* (information update) can be understood in terms of the elimination of possibilities from an agent's initial epistemic state (R. Stalnaker, *Inquiry*, 1987).

Idea 1: Knowledge Operators as Restricted Quantifiers

Example (Berkeley and Copenhagen)

Let K_b stand for **agent** b **knows that** and K_c stand for **agent** c **knows that**. Suppose agent b, who lives in Berkeley, knows that agent c lives in Copenhagen. Let r stand for 'it's raining in Copenhagen'. Although b doesn't know whether it's raining in Copenhagen, b knows that c knows whether it's raining there:

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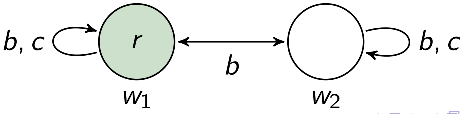
The following picture depicts a situation in which this is true, where an arrow represents *compatibility with one's knowledge*:



Now suppose that agent b doesn't know whether agent c has left Copenhagen for a vacation. (Let v stand for 'c has left Copenhagen on vacation'.) Agent b knows that if c is not on vacation, then c knows whether it's raining in Copenhagen; but if c is on vacation, then c won't bother to follow the weather.

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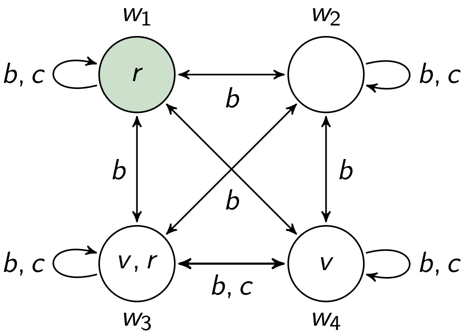

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Definition (Truth)

Given a κ-agent model $\mathcal{M} = \langle W, \{R_a\}_{a \in \mathsf{Agt}}, V \rangle$ with $w \in W$ and $\varphi \in \mathcal{L}_{\mathsf{EL}}^{\kappa}$, we define $\mathcal{M}, w \vDash \varphi$ ("φ is true in $\mathcal{M}$ at w") as follows:

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\mathcal{M}, w \vDash K_a\varphi & \text{iff} & \forall v \in W\text{: if } wR_av \text{ then } \mathcal{M}, v \vDash \varphi;
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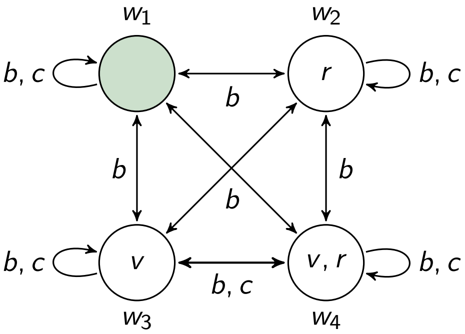

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For language $\mathcal{L}$, the $\mathcal{L}$-theory of a class C of models, $\mathsf{Th}_{\mathcal{L}}(\mathsf{C})$ (resp. class $\mathbb{F}$ of frames) is the set of $\varphi \in \mathcal{L}$ valid on C (resp. $\mathbb{F}$).

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In CS and game theory, it is typically assumed that every R_a is an equivalence relation. We call such models **partition models**.

Complexity

Theorem (Complexity)

1. *model-checking is in linear time (in the size of $\mathcal{M}$ and φ);*
2. *satisfiability in* 1*-agent models is PSPACE-complete;*
3. *satisfiability in* 1*-agent partition models is NP-complete;*
4. *satisfiability in* 2*-agent partition models is PSPACE-complete.*

If we bound the nesting depth of K, satisfiability drops down to NP-complete in most cases; if we further restrict to finitely many atomic sentences, satisfiability drops down to linear time.

Completeness

Theorem

The $\mathcal{L}^{\kappa}_{\mathsf{EL}}$-theory of the class of all κ-agent epistemic models is axiomatized by the system $\mathbf{K}_{\kappa}$:

- *all substitutions of propositional tautologies as axioms;*
- *the K axiom, $K_a(\varphi \to \psi) \to (K_a\varphi \to K_a\psi)$, for $a \in \mathrm{Agt}$;*
- *modus ponens and necessitation, $\vdash \varphi \Rightarrow \vdash K_a\varphi$, for $a \in \mathrm{Agt}$.*

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To axiomatize the theories of other model classes, add the axioms corresponding to the defining properties of the class:

- *for reflexive models, add the T axiom, $K_a\varphi \to \varphi$;*
- *for transitive models, add the 4 axiom, $K_a\varphi \to K_aK_a\varphi$;*
- *for Euclidean models, add the 5 axiom, $\neg K\varphi \to K\neg K\varphi_a$;*
- *etc.*

Idea 2: Common Knowledge as Transitive Closure

The "everyone knows" operator is not the only group knowledge operator of interest. To see this, consider the following example.

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Two agents, b and g, are facing each other. They each see a number on the other's forehead. A perfectly trustworthy source tells them: one of you has $n \in \mathbb{N}$ and the other has $n+1$.

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Let b_m stand for "the number on b's head is m," and let g_m stand for "the number on g's head is m." The truth is: $b_2 \wedge g_3$.

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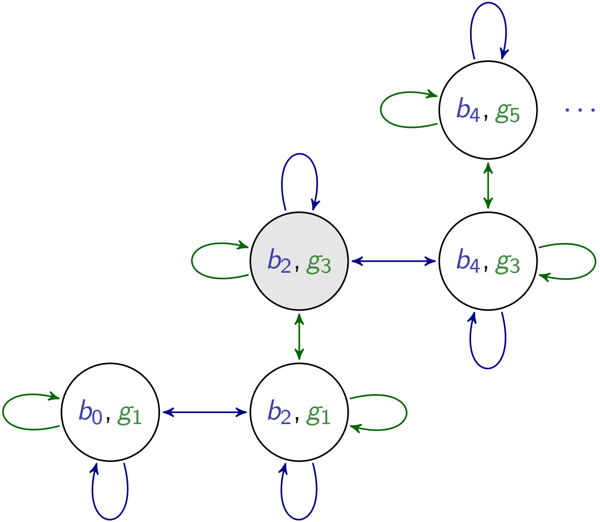
What should an epistemic model representing this look like?

Here is an epistemic model representing the example:



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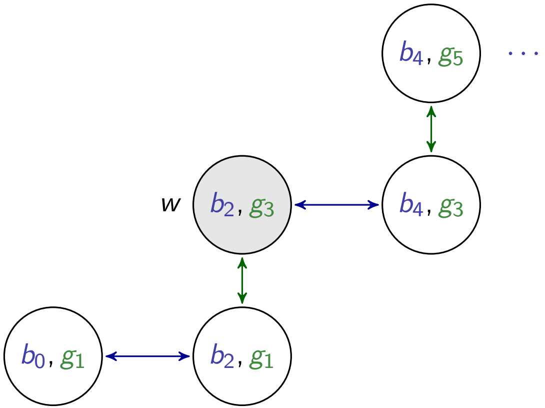

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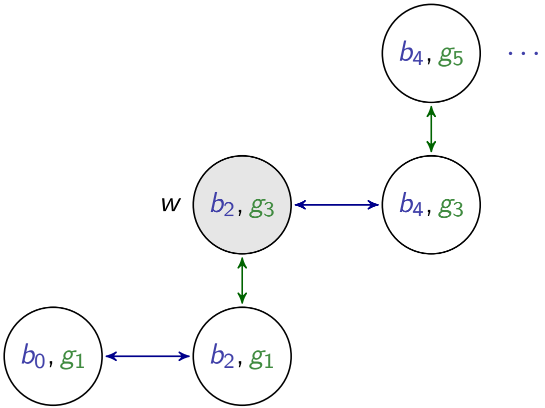

$\mathcal{M}, w \vDash K_b g_3$;

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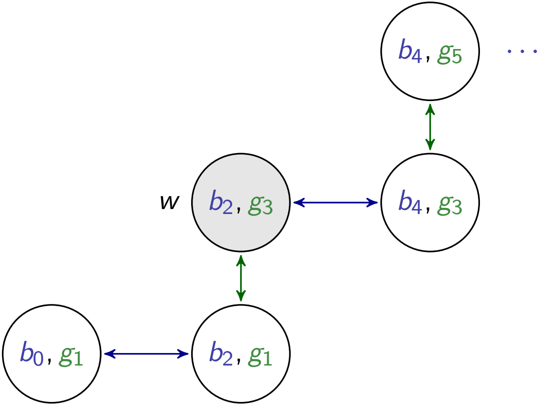

$\mathcal{M}, w \vDash K_b g_3$; $\mathcal{M}, w \vDash K_b(b_2 \vee b_4)$;

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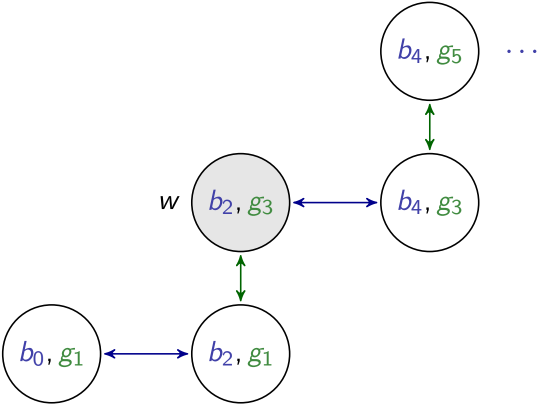

$\mathcal{M}, w \vDash K_b g_3$; $\mathcal{M}, w \vDash K_b(b_2 \vee b_4)$; $\mathcal{M}, w \vDash K_b K_g(g_1 \vee g_3 \vee g_5)$;

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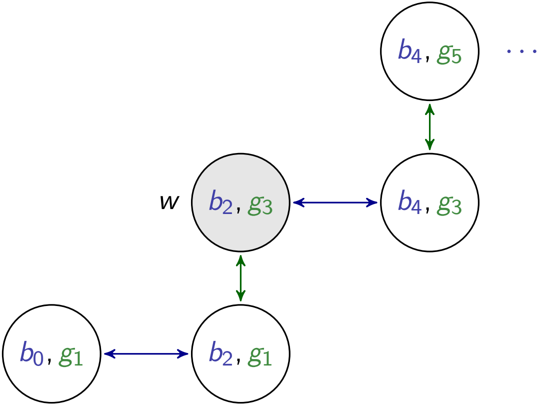

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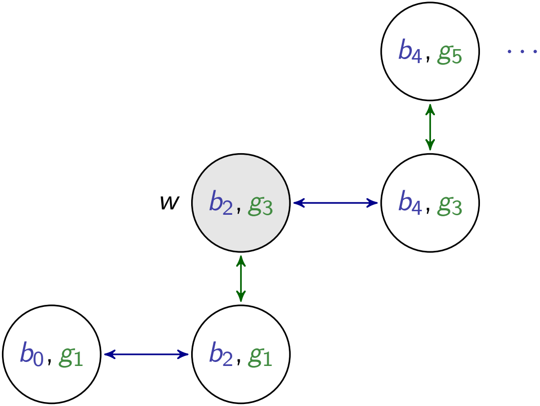

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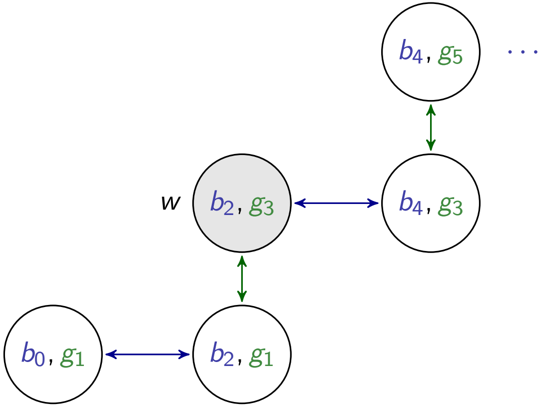

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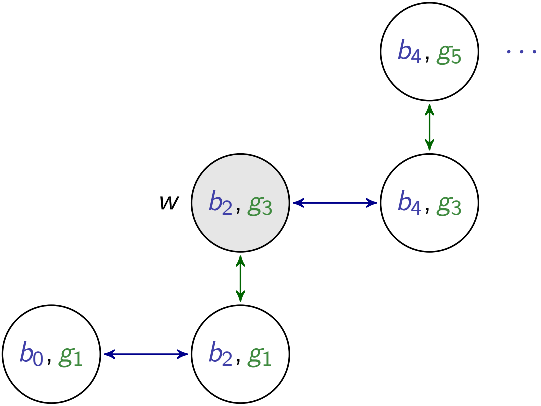

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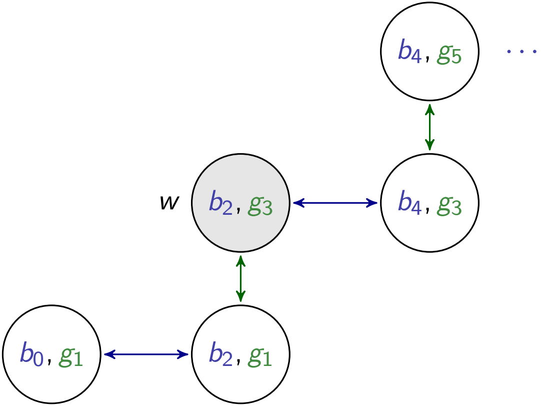

$\mathcal{M}, w \vDash \hat{K}_g g_1 \wedge \hat{K}_g g_3$; $\mathcal{M}, w \vDash \hat{K}_g \hat{K}_b K_g g_1$; and interestingly, $\mathcal{M}, w \vDash E_{\{b,g\}} \neg g_5 \wedge \neg E_{\{b,g\}} E_{\{b,g\}} \neg g_5$. Not common knowledge.

Definition (Basic Epistemic Languages)

Given countable sets $\mathsf{At} = \{p, q, r, \dots\}$ and $\mathsf{Agt} = \{a, b, c, \dots\}$ with $|\mathsf{Agt}| = \kappa$, the epistemic language $\mathcal{L}^{\kappa}_{\mathsf{EL}}$ is generated by

$$\varphi ::= p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid K_a\varphi.$$

For the epistemic language with common knowledge, $\mathcal{L}^{\kappa}_{\mathsf{EL\text{-}C}}$,

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where $A \subseteq \mathsf{Agt}$.

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where $A \subseteq \mathsf{Agt}$. We read $C_A\varphi$ as "**it is common knowledge among the members of** A **that** φ."

Idea 2: Common Knowledge as Transitive Closure

Definition

Given a κ-agent model $\mathcal{M} = \langle W, \{R_a\}_{a \in \mathsf{Agt}}, V \rangle$ with $w \in W$ and $\varphi \in \mathcal{L}^{\kappa}_{\mathsf{EL\text{-}C}}$, we define $\mathcal{M}, w \vDash \varphi$ as follows (other cases as before):

$$\mathcal{M}, w \vDash K_a\varphi \quad \text{iff} \quad \forall v \in W\colon wR_av \text{ implies } \mathcal{M}, v \vDash \varphi;$$

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where R_A^{+} is the transitive closure of $\bigcup_{a \in A} R_a$.

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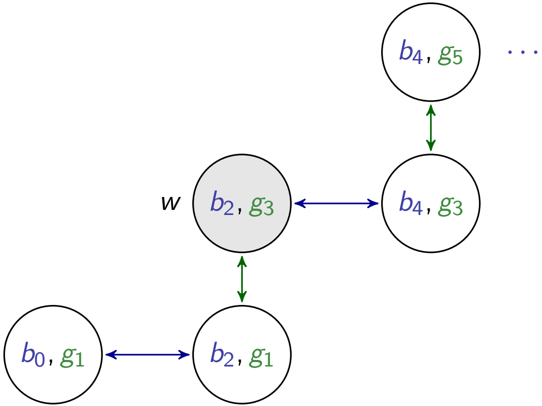
where R^+_A is the transitive closure of $\bigcup_{a \in A} R_a$.

In other words, $C_A\varphi$ is true at w iff any sequence of steps from w along any agents' relations ends in a state where φ is true.

Example: Consecutive Numbers



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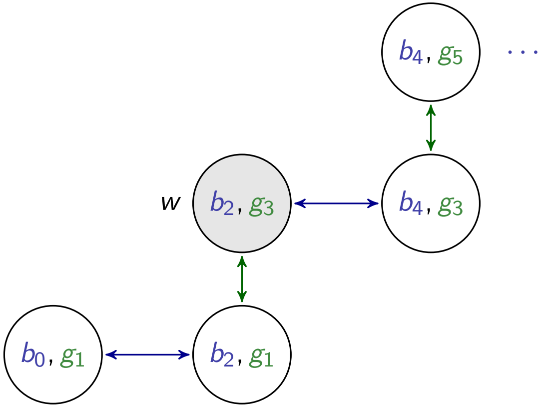

Recall: $\mathcal{M}, w \vDash E_{\{b,g\}}\neg g_5 \wedge \neg E_{\{b,g\}}E_{\{b,g\}}\neg g_5$.

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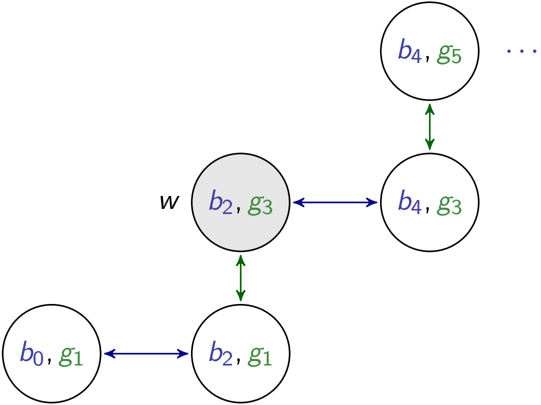
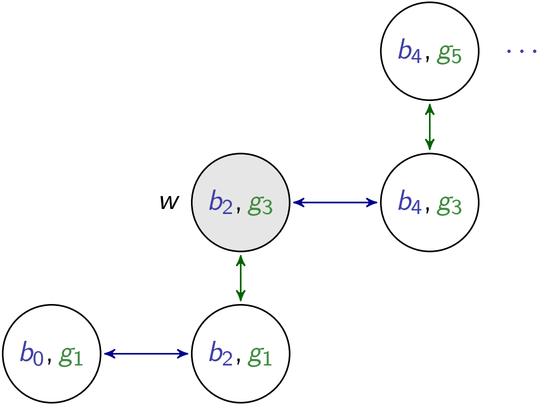


Recall: $\mathcal{M}, w \vDash E_{\{b,g\}} \neg g_5 \wedge \neg E_{\{b,g\}} E_{\{b,g\}} \neg g_5$.

Pattern: $\mathcal{M}, w \vDash E^3_{\{b,g\}} \neg g_6 \wedge \neg E^4_{\{b,g\}} \neg g_6$.

Example: Consecutive Numbers

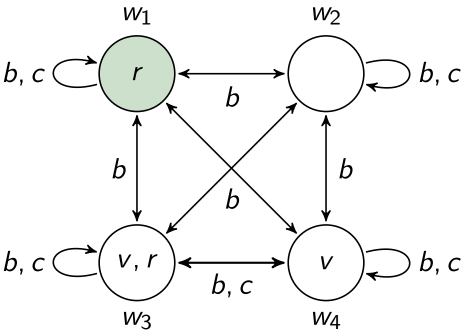


Surprising result: $\mathcal{M}, w \vDash K_b g_3 \wedge K_g b_2 \wedge \neg C_{\{b,g\}} \neg b_{1000}$.

It's not common knowledge that b doesn't have 1000 on his head!

By contrast, in our earlier example with Berkeley and Copenhagen, it is common knowledge that if c is on vacation, then he doesn't know whether or not it's raining in Copenhagen:

$$\mathcal{M}, w_1 \vDash C_{\{b,c\}}(v \to \neg(K_c r \vee K_c \neg r)).$$



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where R^+_A is the transitive closure of $\bigcup_{a \in G} R_a$.

Observation

Given the infinitary character of common knowledge, it's no surprise that with $\mathcal{L}_{\mathsf{EL\text{-}C}}$ we lose compactness:

$$\{E^n_A p \mid n \in \mathbb{N}\} \cup \{\neg C_A p\}$$

is finitely-satisfiable but not satisfiable.

Completeness and Complexity

Theorem

The $\mathcal{L}_{\mathsf{EL\text{-}C}}$-theory of the class of all (resp. reflexive, preorder, partition, etc.) models is axiomatized by $\mathbf{K}_\kappa$ *(resp.* $\mathbf{T}_\kappa$, $\mathbf{S4}_\kappa$, $\mathbf{S5}_\kappa$, *etc.) plus the following schemas for all* $A \subseteq \mathrm{Agt}$*:*

- *Fixed-Point Axiom:* $C_A\varphi \leftrightarrow (\varphi \wedge E_A C_A \varphi)$

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Complexity of checking satisfiability goes to EXPTIME-complete with common knowledge.

Idea 3: Information Update as Model Restriction

Example (Muddle Children Puzzle)

Three children—r, g, and b—have returned after playing outside. Their mother says to them, "At least one of you has mud on your forehead. Step forward now if you know whether you are dirty."

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None of the children step forward. So their mother says again, “Step forward now if you know whether you are dirty.”

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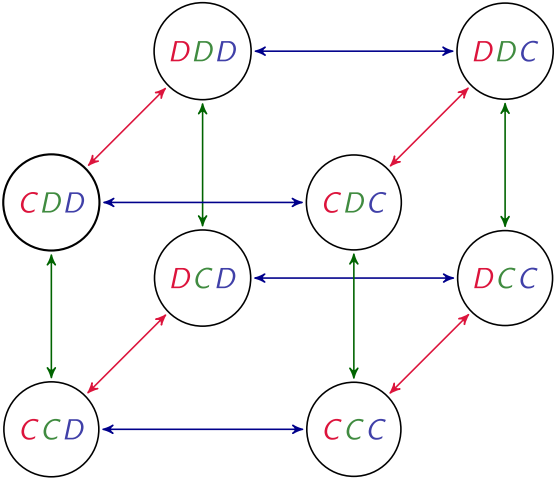
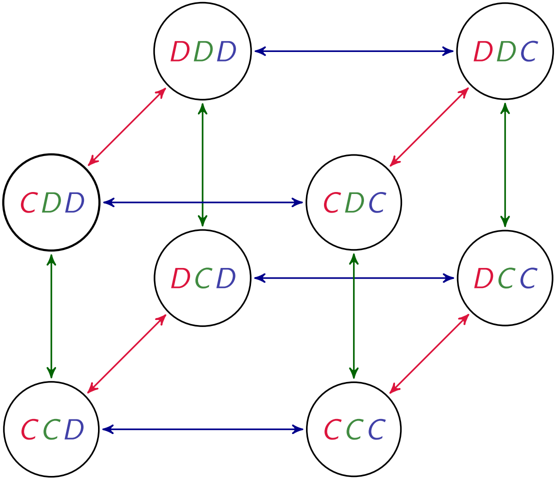
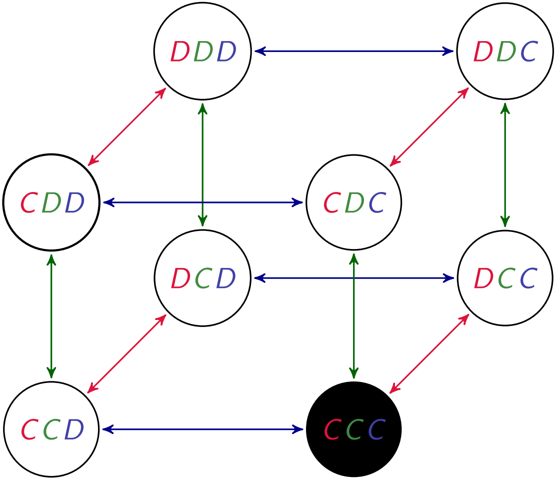
So their mother says again. “Step forward now if you know whether you are dirty.” Finally r steps forward.


DDD
DDC
CDD
CDC
DCD
DCC
CCD
CCC



"At least one of you is dirty." $D_r \vee D_g \vee D_b$. What happens?



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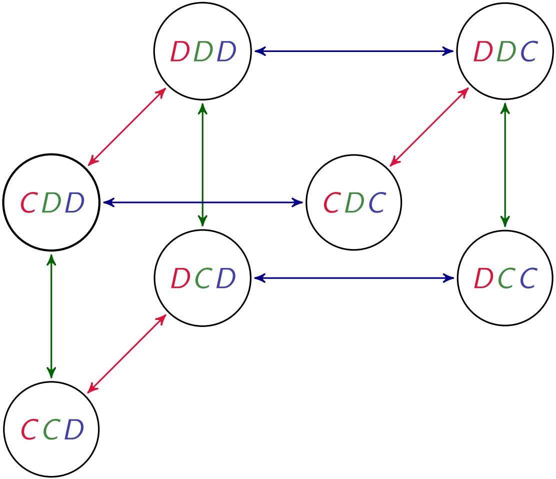

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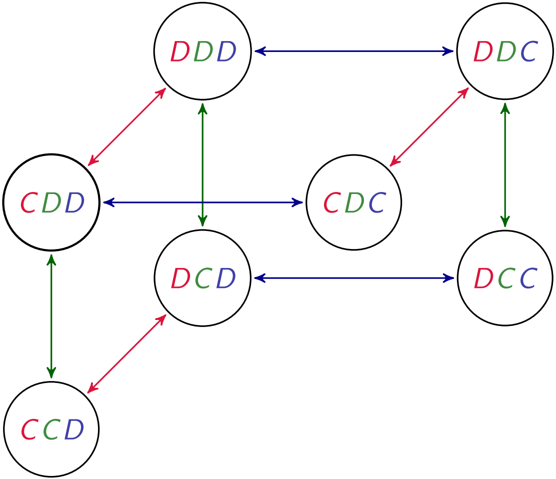
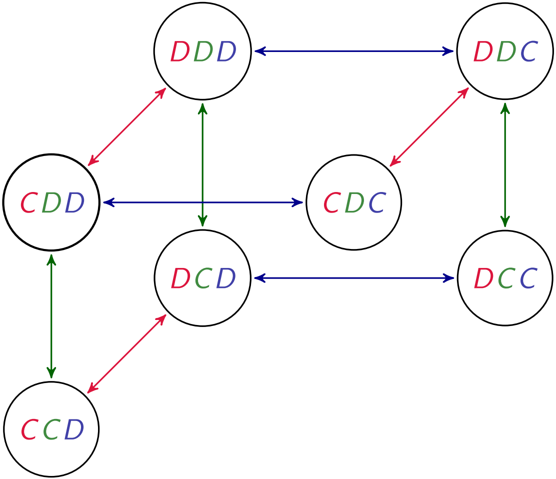
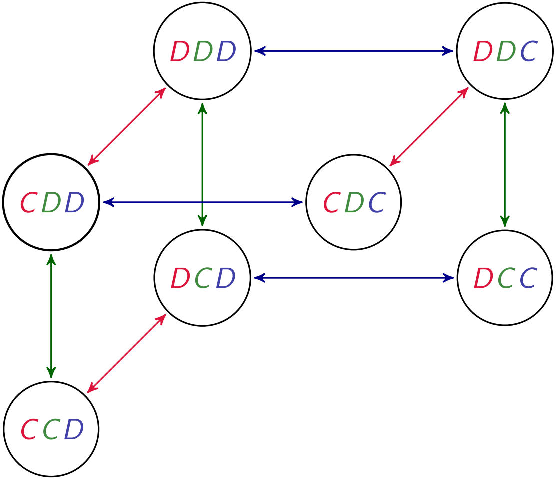


"If you know whether you're dirty, step forward."



"If you know whether you're dirty, step forward." No one does.
$\neg(K_r D_r \vee K_r \neg D_r) \wedge \neg(K_g D_g \vee K_g \neg D_g) \wedge \neg(K_b D_b \vee K_b \neg D_b)$.



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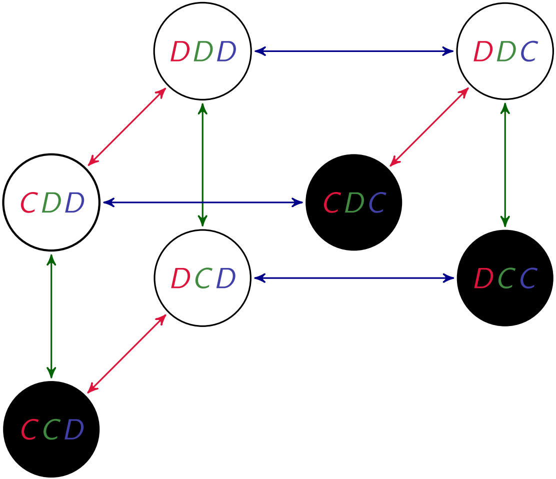
What happens after this is announced?



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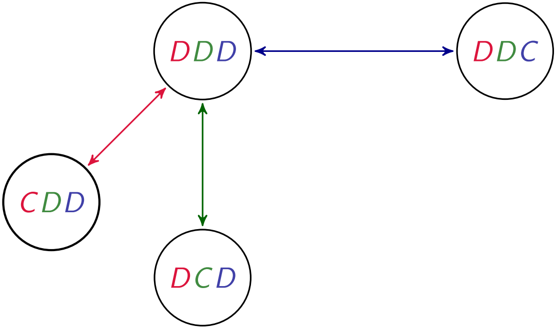
What happens after this? Three more possibilities are eliminated.



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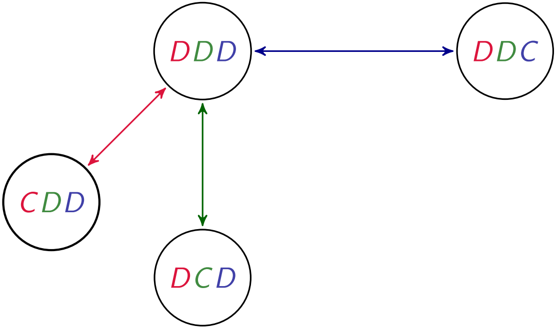
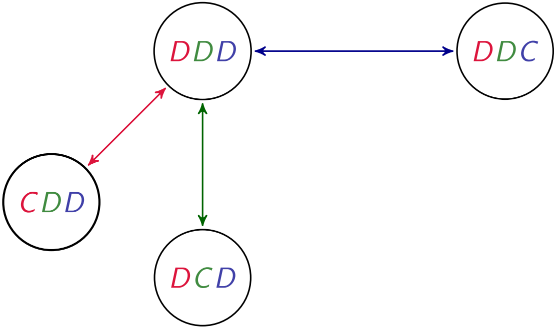


"If you know whether you're dirty, step forward." Blue and green (but not red) step forward.



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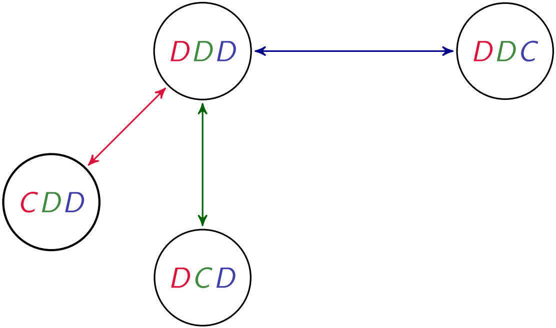

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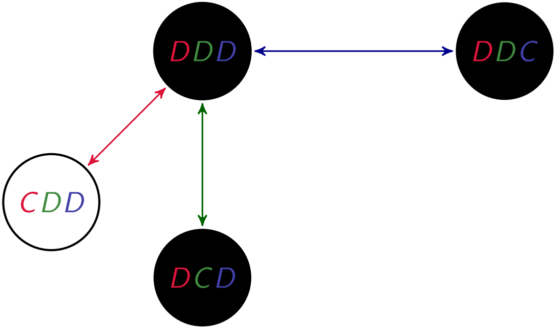

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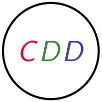
The remaining non-actual possibilities are eliminated.



$(K_b D_b \vee K_b \neg D_b) \wedge (K_g D_g \vee K_g \neg D_g)$. What happens after this?
The remaining non-actual possibilities are eliminated.

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One can also add common knowledge C_A to the language.

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For the epistemic language with common knowledge, $\mathcal{L}^{\kappa}_{\mathsf{EL\text{-}C}}$,

$$\varphi ::= p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid K_a\varphi \mid C\varphi.$$

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For public announcement with common knowledge, $\mathcal{L}^{\kappa}_{\mathsf{PAL\text{-}C}}$,

$$\varphi ::= p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid K_a\varphi \mid [\varphi]\varphi \mid C\varphi.$$

Idea 3: Information Update as Model Restriction

Definition (Truth cont.)

Given a κ-agent model $\mathcal{M} = \langle W, \{R_a\}_{a \in \mathsf{Agt}}, V\rangle$ with $w \in W$ and $\varphi \in \mathcal{L}^{\kappa}_{\mathsf{PAL}}$, we define $\mathcal{M}, w \vDash \varphi$ as follows (other cases as before):

$$\mathcal{M}, w \vDash [\varphi]\psi \quad \text{iff} \quad \mathcal{M}, w \vDash \varphi \text{ implies } \mathcal{M}_{|\varphi}, w \vDash \psi,$$

where $\mathcal{M}_{|\varphi} = \langle W_{|\varphi}, \{R_{a_{|\varphi}}\}_{a \in \mathsf{Agt}}, V_{|\varphi}\rangle$ is the submodel such that

$$W_{|\varphi} = \{v \in W \mid \mathcal{M}, v \vDash \varphi\};$$
$$\forall a \in \mathsf{Agt}\colon R_{a_{|\varphi}} = R_a \cap (W_{|\varphi} \times W_{|\varphi});$$
$$\forall p \in \mathsf{At}\colon V_{|\varphi}(p) = V(p) \cap W_{|\varphi}.$$

Public Announcement Logic

Read $[\varphi]\psi$ as "after (every) true announcement of φ, ψ."

Read $\langle\varphi\rangle\psi := \neg[\varphi]\neg\psi$ as "after a true announcement of φ, ψ."

Let's compare. The truth clause for the dynamic operator $[\varphi]$ is:

- $\mathcal{M}, w \vDash [\varphi]\psi$ iff $\mathcal{M}, w \vDash \varphi$ *implies* $\mathcal{M}_{|\varphi}, w \vDash \psi$.

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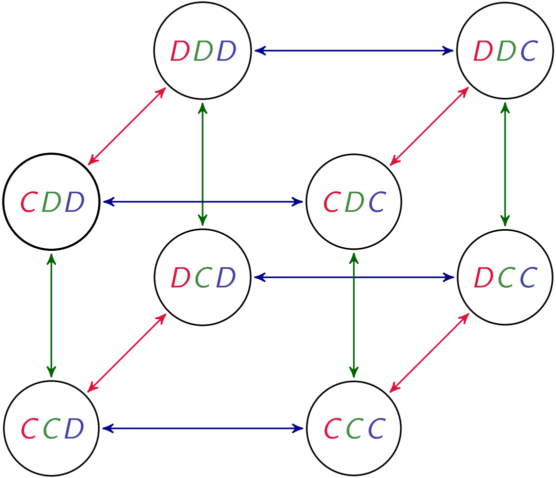
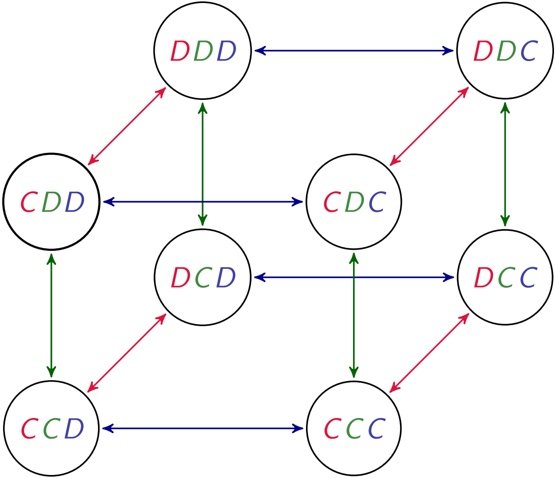
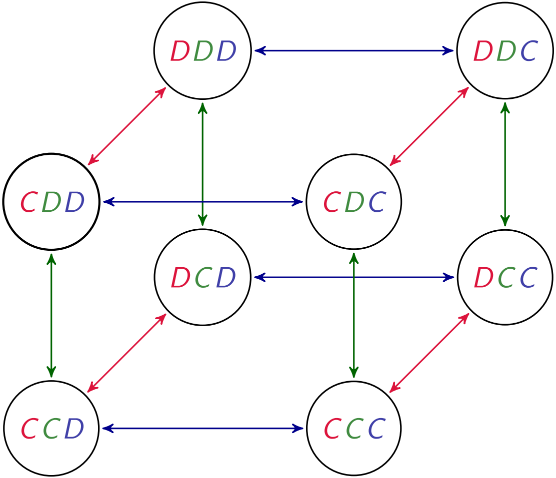
Big Idea: we evaluate $[\varphi]\psi$ and $\langle\varphi\rangle\psi$ not by looking at *other worlds in the same model*, but rather by looking at a new model.



Where $A_1 := D_r \vee D_g \vee D_b$, check $\mathcal{M}, CCD \vDash \langle A_1 \rangle K_b D_b$.

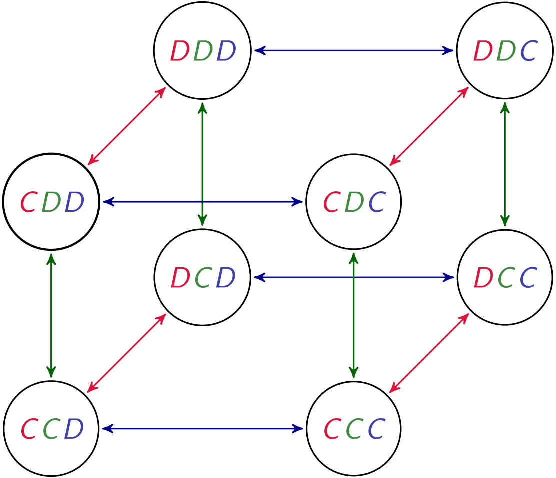


Where $A_1 := D_r \vee D_g \vee D_b$, check $\mathcal{M}, CCD \vDash \langle A_1 \rangle K_b D_b$.
A_1 is Mom's announcement: "At least one of you is dirty."



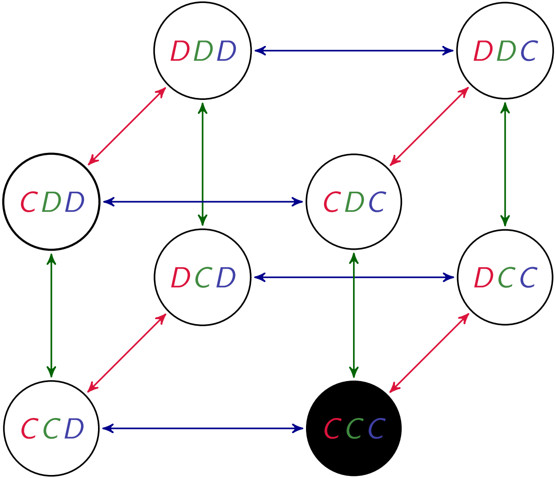
Where $A_1 := D_r \vee D_g \vee D_b$, check $\mathcal{M}, CCD \vDash \langle A_1 \rangle K_b D_b$.

First, observe that $\mathcal{M}, CCD \vDash D_r \vee D_g \vee D_b$.



Where $A_1 := D_r \vee D_g \vee D_b$, check $\mathcal{M}, CCD \vDash \langle A_1 \rangle K_b D_b$.

Second, delete all worlds where $D_r \vee D_g \vee D_b$ is false.



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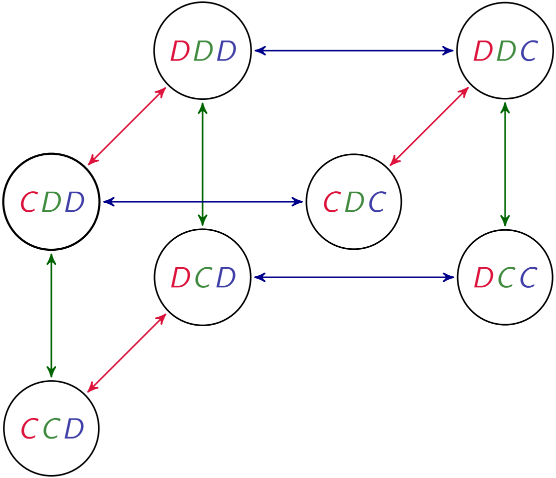
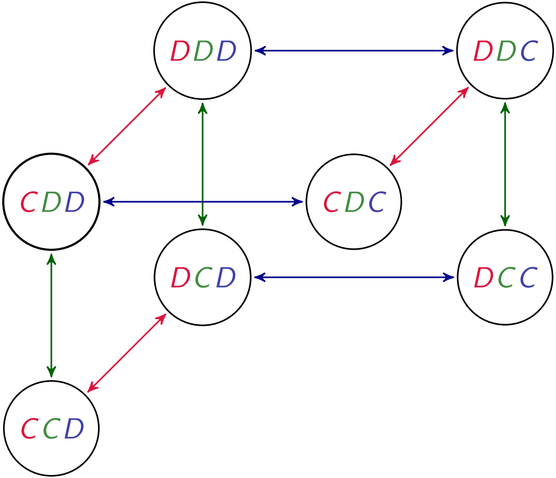
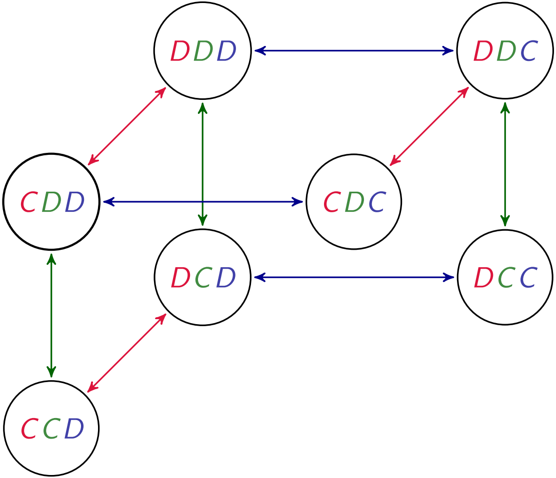


Where $A_1 := D_r \vee D_g \vee D_b$, check $\mathcal{M}, CCD \vDash \langle A_1 \rangle K_b D_b$.

Finally, observe that $\mathcal{M}_{|A_1}, CCD \vDash K_b D_b$.

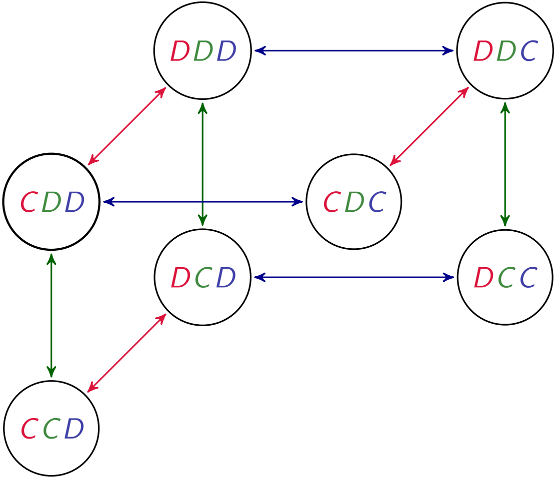


Where $A_2 := \bigwedge_{a \in \{r,g,b\}} \neg(K_a D_a \vee K_a \neg D_a)$, representing that no one steps forward, check $\mathcal{M}_{|A_1}, CDD \vDash \langle A_2 \rangle (K_g D_g \wedge K_b D_b)$.



$\mathcal{M}_{|A_1}, CDD \vDash \langle \bigwedge_{a \in \{r,g,b\}} \neg(K_a D_a \vee K_a \neg D_a) \rangle (K_g D_g \wedge K_b D_b)$.

First, note $\mathcal{M}_{|A_1}, CDD \vDash \bigwedge_{a \in \{r,g,b\}} \neg(K_a D_a \vee K_a \neg D_a)$.



$$\mathcal{M}_{|A_1}, CDD \vDash \langle \bigwedge_{a \in \{r,g,b\}} \neg(K_a D_a \vee K_a \neg D_a) \rangle (K_g D_g \wedge K_b D_b).$$

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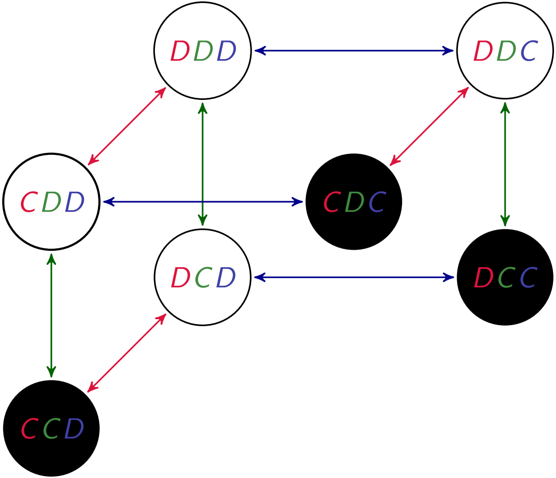

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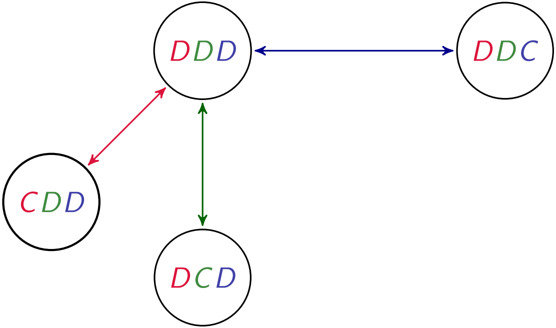
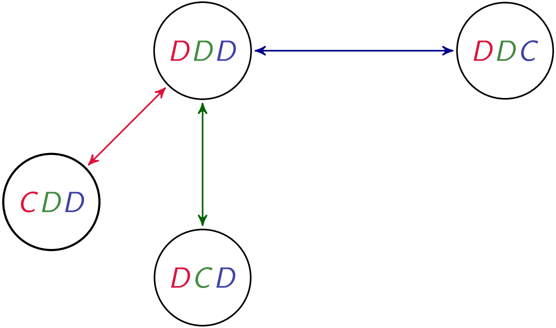
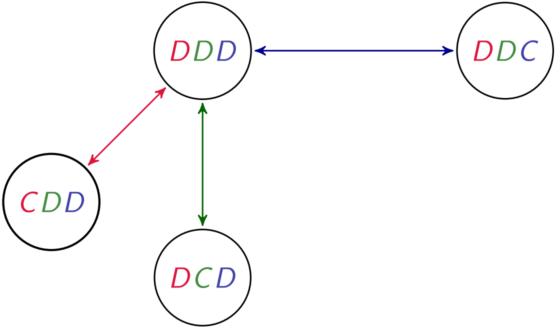


$$\mathcal{M}_{|A_1}, CDD \vDash \langle \bigwedge_{a \in \{r,g,b\}} \neg(K_a D_a \vee K_a \neg D_a) \rangle (K_g D_g \wedge K_b D_b).$$

Finally, observe $\mathcal{M}_{|A_1| \bigwedge_{a \in \{r,g,b\}} \neg(K_a D_a \vee K_a \neg D_a)}, CDD \vDash K_g D_g \wedge K_b D_b$.

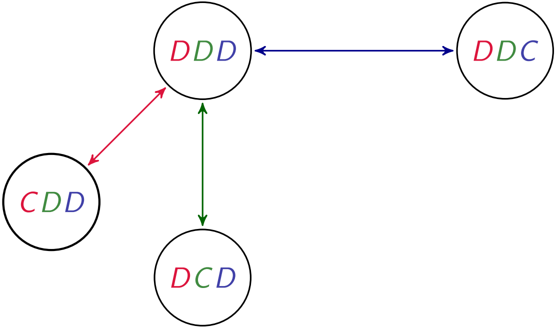


Check $\mathcal{M}_{|A_1|A_2}, CDD \vDash \langle \bigwedge_{a \in \{g,b\}} (K_a D_a \vee K_a \neg D_a) \rangle K_r \neg D_r$.



Check $\mathcal{M}_{|A_1|A_2}, CDD \vDash \langle \bigwedge_{a \in \{g,b\}} (K_a D_a \vee K_a \neg D_a) \rangle K_r \neg D_r$.

This represents the info. given when g and b step forward after Mom says *again*, "If you know whether you're dirty, step forward."



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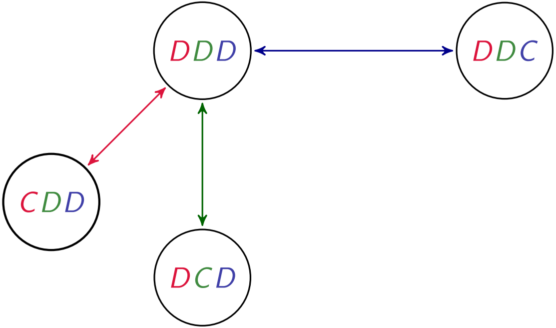

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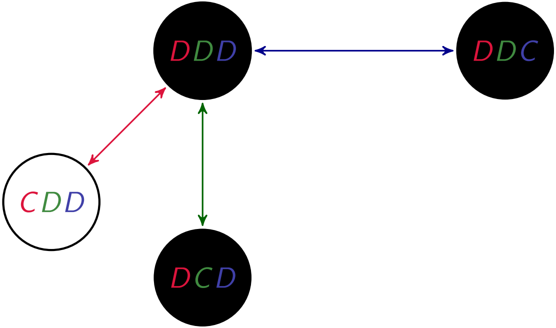

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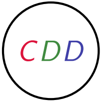

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Finally, observe $\mathcal{M}_{|A_1|A_2| \bigwedge_{a \in \{g,b\}} (K_a D_a \vee K_a \neg D_a)}, CDD \vDash K_r \neg D_r$.

Completeness and Complexity

Theorem (PAL Axiomatization (Plaza 1989))

The $\mathcal{L}^{\kappa}_{\mathsf{PAL}}$-theory of the class of all (resp. reflexive, preorder, partition) models is axiomatized by $\mathbf{K}_\omega$ (resp. $\mathbf{T}_\omega$, $\mathbf{S4}_\omega$, $\mathbf{S5}_\omega$) plus:

i.	(replacement)	$\dfrac{\psi \leftrightarrow \chi}{\varphi(\psi/p) \leftrightarrow \varphi(\chi/p)}$
ii.	(atomic reduction)	$\langle\varphi\rangle p \leftrightarrow (\varphi \wedge p)$
iii.	(negation reduction)	$\langle\varphi\rangle\neg\psi \leftrightarrow (\varphi \wedge \neg\langle\varphi\rangle\psi)$
iv.	(conjunction reduction)	$\langle\varphi\rangle(\psi \wedge \chi) \leftrightarrow (\langle\varphi\rangle\psi \wedge \langle\varphi\rangle\chi)$
v.	(diamond reduction)	$\langle\varphi\rangle\hat{K}_a\psi \leftrightarrow (\varphi \wedge \hat{K}_a\langle\varphi\rangle\psi)$.

Complexity of satisfiability for PAL is the same as for EL.

With the background of basic epistemic logic in place, let's turn to mathematical themes arising in recent work in epistemic logic:

- Preservation Theorems
- Uniform Substitution
- Relations to FOL

Preservation of Truth (in Inquiry)

Definition (Preservation Under Submodels)

A formula φ is preserved under submodels iff for any pointed model $\mathcal{M}, w$ and $\mathcal{M}' \subseteq \mathcal{M}$: $\mathcal{M}, w \vDash \varphi \Rightarrow \mathcal{M}', w \vDash \varphi$.

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Given our epistemic understanding of the models, this kind of preservation takes on a new meaning: once true, φ remains true henceforth in inquiry, no matter what new knowledge is acquired.

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The universal fragment of $\mathcal{L}^{\kappa}_{\mathsf{EL}}$ is generated by:

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Theorem (Modal Łoś-Tarski)

A formula $\varphi \in \mathcal{L}_{\mathsf{EL}}^{\kappa}$ is preserved under submodels **iff** *it is equivalent to a formula in the universal fragment of $\mathcal{L}_{\mathsf{EL}}^{\kappa}$.*

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The universal fragment of $\mathcal{L}^{\kappa}_{\text{PAL-C}}$ is generated by:

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Theorem (van Ditmarsch, Kooi)

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Open Problem 1: does 'only if' hold as well?

Definition (Preservation Under Submodels)

A formula φ is preserved under submodels iff for any pointed model $\mathcal{M}, w$ and $\mathcal{M}' \subseteq \mathcal{M}$: $\mathcal{M}, w \vDash \varphi \Rightarrow \mathcal{M}', w \vDash \varphi$.

Open Problem 1: Which formulas of $\mathcal{L}_{\text{EL-C}}$ and $\mathcal{L}_{\text{PAL-C}}$ (with common knowledge), syntactically characterized, are preserved?

(A place to start might be the proof of Łoś-Tarski for the μ-calculus (D'Agostino, 1997), of which $\mathcal{L}_{\text{EL-C}}$ is a fragment.)

Definition (Successful Formulas)

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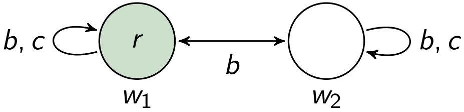
The simplest example: the Copenhagen agent c calls the Berkeley agent b on the phone and says "You don't know it's raining in Copenhagen, but it's raining in Copenhagen" ($\neg K_b r \wedge r$).



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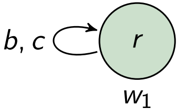
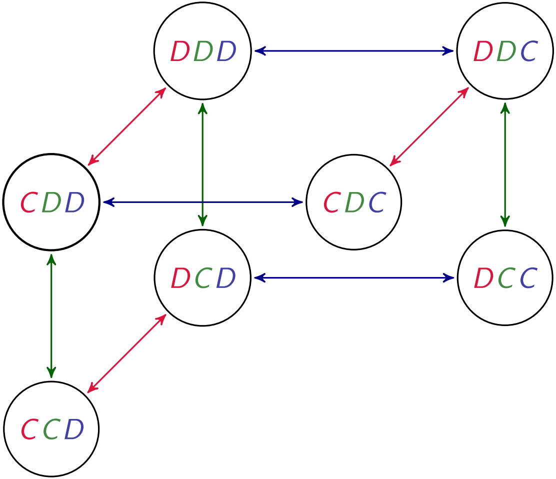
Not all formulas are successful. Some formulas are *unsuccessful*.

The simplest example: the Copenhagen agent c calls the Berkeley agent b on the phone and says "You don't know it's raining in Copenhagen, but it's raining in Copenhagen" ($\neg K_b r \wedge r$).



Let's go back to the first time that no students step forward. This is like their announcing, "We don't know whether we are dirty."



$\mathcal{M}_{|A_1}, CDD \vDash \langle \bigwedge_{a \in \{r,g,b\}} \neg(K_a D_a \vee K_a \neg D_a) \rangle (K_g D_g \wedge K_b D_b)$.

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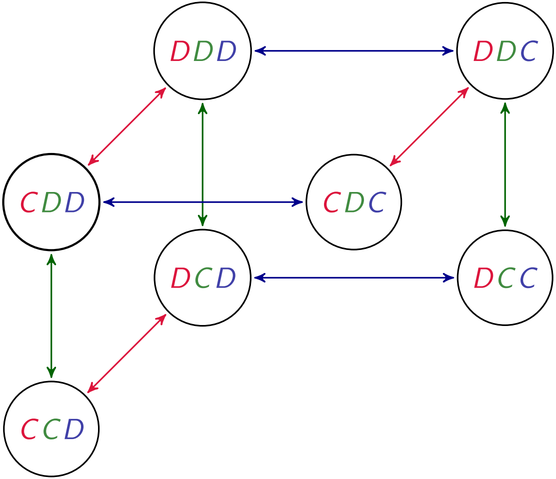

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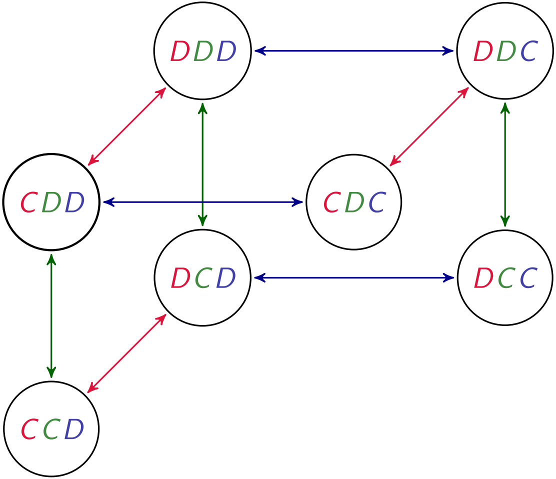
First, note $\mathcal{M}, CDD \vDash \bigwedge_{a \in \{r,g,b\}} \neg(K_a D_a \vee K_a \neg D_a)$.



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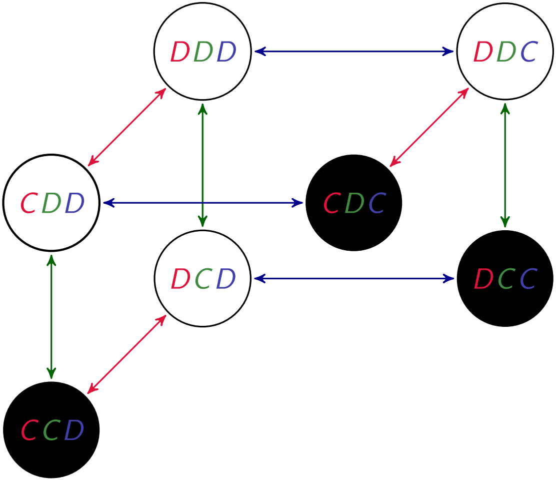
Second, delete all worlds where $\bigwedge_{a \in \{r,g,b\}} \neg(K_a D_a \vee K_a \neg D_a)$ is false.



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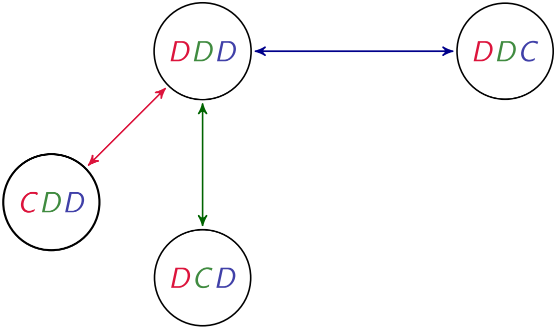

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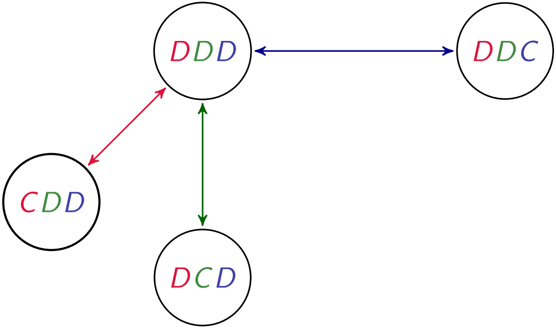
Finally, observe $\mathcal{M}_{|A_1| \bigwedge_{a \in \{r,g,b\}} \neg(K_a D_a \vee K_a \neg D_a)}, CDD \vDash K_g D_g \wedge K_b D_b$.



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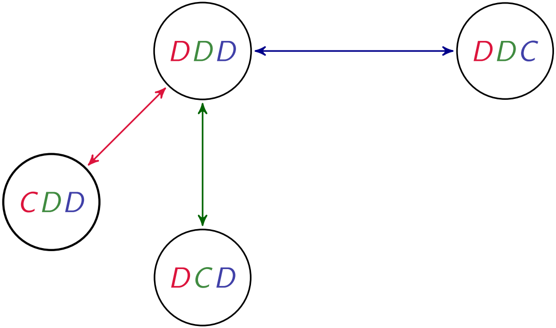
Hence where $\varphi := \bigwedge_{a\in\{r,g,b\}} \neg(K_aD_a \vee K_a\neg D_a)$,

$$\mathcal{M}_{|A_1}, CDD \vDash \langle\varphi\rangle\neg\varphi.$$



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The childrens' **true** announcement that no one knows makes it **false** that no one knows! This is a so-called *unsuccessful* update.

Definition (Successful Formulas)

A formula φ is successful iff for all pointed models $\mathcal{M}, w$,

$$\mathcal{M}, w \vDash \varphi \Rightarrow \mathcal{M}_{|\varphi}, w \vDash \varphi.$$

Successful formulas are preserved under "self-defined" submodels.

Not all formulas are successful. Some formulas are *unsuccessful*.

Example (Unsuccessful Formulas)

We've seen two examples so far:

- $\neg K_b r \wedge r$
- $\bigwedge_{a \in \{r,g,b\}} \neg(K_a D_a \vee K_a \neg D_a)$

Definition (Successful Formulas)

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Which formulas of $\mathcal{L}_{\mathsf{EL}}^{\mathcal{K}}$, syntactically characterized, are successful over the class of all models? The class of partition models?

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The problem of deciding whether a formula $\varphi \in \mathcal{L}_{\mathsf{EL}}^{1}$ is successful over the class of partition models is co-NP complete.

So we cannot expect a very simple syntactic characterization.

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In essence, we show that all *un*successful formulas involve variations on the Moorean theme of "you don't know it, but *p*."

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In essence, we show that all *un*successful formulas involve variations on the Moorean theme of "you don't know it, but *p*."

Open Problem 2: answer the question for $\kappa > 1$ or for non-partition models (cf. Saraf and Sourabh for some leads).

Other Interesting Formula Classes

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A formula φ is self-refuting iff for all pointed models $\mathcal{M}, w$,

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Define $\mathcal{M}_{|^n\varphi}$ recursively by $\mathcal{M}_{|^0\varphi} = \mathcal{M}$, $\mathcal{M}_{|^{n+1}\varphi} = \left(\mathcal{M}_{|^n\varphi}\right)_{|\varphi}$.
φ is eventually self-refuting iff for all pointed models $\mathcal{M}, w$:

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Definition (Always Learnable)

φ is always learnable (for a) iff for any pointed model $\mathcal{M}, w$:

$$\mathcal{M}, w \vDash \varphi \Rightarrow \exists \psi : \mathcal{M}_{|\psi}, w \vDash K_a\varphi.$$

Substitution Closure

A striking feature of epistemic logic with operators for information update is that it is not closed under uniform substitution: e.g., $[p]p$ is valid, but $[\neg K_b r \wedge r](\neg K_b r \wedge r)$ is not valid, as we saw.

Definition (Substitution Core & Schematic Validity)

A substitution is any function $\sigma\colon \mathsf{At} \to \mathcal{L}$; and we define $(\cdot)^\sigma\colon \mathcal{L} \to \mathcal{L}$ as the extension such that $(\varphi)^\sigma$ is obtained from φ by replacing, for all $p \in \mathsf{At}(\varphi)$, each occurrence of p in φ by $\sigma(p)$.

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The substitution core of logic L is the set

$$\{\varphi \in \mathcal{L}_{\mathsf{L}} \mid (\varphi)^\sigma \in \mathsf{L} \text{ for all substitutions } \sigma\}.$$

Formulas in the substitution core are schematically valid.

Example (Valid but not Schematically Valid)

Where C is the class of all models, formulas that are in $\mathrm{Th}_{\mathcal{L}^{\kappa}_{\mathrm{PAL}}}(C)$ but are not in the substitution core of $\mathrm{Th}_{\mathcal{L}^{\kappa}_{\mathrm{PAL}}}(C)$ include ($\kappa \geq 1$):

$$[p]p \qquad K_a p \to [p]K_a p$$
$$[p]K_a p \qquad K_a p \to [p](p \to K_a p)$$
$$[p](p \to K_a p) \qquad K_a(p \to q) \to (\langle q\rangle K_a r \to \langle p\rangle K_a r)$$
$$[p \wedge \neg K_a p]\neg(p \wedge \neg K_a p) \qquad (\langle p\rangle K_a r \wedge \langle q\rangle K_a r) \to \langle p \vee q\rangle K_a r.$$

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$[p]p$	$K_a p \to [p]K_a p$
$[p]K_a p$	$K_a p \to [p](p \to K_a p)$
$[p](p \to K_a p)$	$K_a(p \to q) \to (\langle q \rangle K_a r \to \langle p \rangle K_a r)$
$[p \wedge \neg K_a p]\neg(p \wedge \neg K_a p)$	$(\langle p \rangle K_a r \wedge \langle q \rangle K_a r) \to \langle p \vee q \rangle K_a r.$

So what are the uniform principles of information dynamics?

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Theorem (Axiomatization)

For the class C of all (resp. reflexive, preorder, partition) ω-agent models, the substitution core of $Th_{\mathcal{L}^{\omega}_{\mathsf{PAL}}}(C)$ is axiomatizable.

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Upshot: we have a logic for reasoning about information change in which the atomic sentences p can be genuine propositional variables, standing in for any propositions, even epistemic ones.

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10.	(ap-commutativity)	$\hat{K}_a\langle p\rangle q \to [p]\hat{K}_a q$
11.	(composition)	$\langle p\rangle\langle q\rangle r \leftrightarrow \langle\langle p\rangle q\rangle r$

Theorem (Axiomatization)

For the class C of all (resp. reflexive, preorder, partition) ω-agent models, the substitution core of $Th_{\mathcal{L}^{\omega}_{\mathsf{PAL}}}(C)$ is axiomatizable.

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Wesley Holliday, Tomohiro Hoshi, and Thomas Icard. Forthcoming.
"Information Dynamics and Uniform Substitution." *Synthese*.

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"Schematic Validity in Dynamic Epistemic Logic: Decidability." *LNAI*.

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(The obstacle is that given the failure of compactness, we must use a finite version of the Henkin-style canonical model construction to prove completeness; but the use of the standard Fisher-Ladner closure to do so proves difficult in this case...)

Transferring the Questions to FOL

In "Open Problems in Logical Dynamics," van Benthem poses the questions we have been discussing for FOL as well.

Open Problem 4: Characterize the formulas φ of FOL with exactly one free variable x such that for any FO structure $\mathfrak{A}$ and variable assignment s, if $\mathfrak{A}, s \vDash \varphi$, then $\mathfrak{A}_\varphi, s \vDash \varphi$, where $\mathfrak{A}_\varphi$ is the substructure of $\mathfrak{A}$ with domain $\{d \in |\mathfrak{A}| \mid \mathfrak{A}, s_{[x:=d]} \vDash \varphi\}$.

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Open Problem 5: Extend the language of FOL with relativization operators $(\cdot)^\varphi$ for all φ with exactly one free variable x, such that

$$\mathfrak{A}, s \vDash (\psi)^\varphi \Leftrightarrow \text{if } \forall y \in FV(\psi)\, \mathfrak{A}, s_{[x:=s(y)]} \vDash \varphi, \text{ then } \mathfrak{A}_\varphi, s \vDash \psi,$$

where $FV(\psi)$ is the set of free variables in ψ.

What is the complete logic of FOL with relativization?

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Open Problem 5: Extend the language of FOL with relativization operators $(\cdot)^{\varphi}$ for all φ with exactly one free variable x, such that

$$\mathfrak{A}, s \vDash (\psi)^{\varphi} \Leftrightarrow \text{if } \forall y \in FV(\psi)\, \mathfrak{A}, s_{[x:=s(y)]} \vDash \varphi, \text{ then } \mathfrak{A}_{\varphi}, s \vDash \psi,$$

where $FV(\psi)$ is the set of free variables in ψ.

What is the complete logic of FOL with relativization?

Since this logic is not closed under uniform substitution (e.g., $Px \rightarrow (Px)^{Qz}$ is valid, but $(Px \wedge \exists y \neg Py) \rightarrow (Px \wedge \exists y \neg Py)^{Qz}$ is not), is the substitution core of this logic axiomatizable?

Translating EL into FOL

Can we transfer our results for modal logic to fragments of FOL?

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Definition (Propositional Epistemic Language)

Given sets $\text{At} = \{p, q, r, \dots\}$ and $\text{Agt} = \{a, b, c, \dots\}$ with $|\text{Agt}| = \kappa$, the epistemic language $\mathcal{L}^{\kappa}_{\text{EL}}$ is generated by

$$\varphi ::= p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid K_a\varphi.$$

Consider a first-order language that contains, for every $p \in \text{At}$ and $a \in \text{Agt}$, a one-place predicate $\mathbf{P}$ and a two-place predicate $\mathbf{R}_a$.

Translating EL into FOL

Definition (Standard Translation)

Fix two variables x and y and define functions ST_x and ST_y:

- $ST_x(p) = \mathbf{P}x$;
- $ST_x(\neg\varphi) = \neg ST_x(\varphi)$;
- $ST_x(\varphi \wedge \psi) = ST_x(\varphi) \wedge ST_x(\psi)$;
- $ST_x(K_a\varphi) = \forall y(\mathbf{R}_a xy \to ST_y(\varphi))$

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- and similarly for ST_y.

Any modal model $\mathcal{A}$ can be viewed as a first-order structure $\mathfrak{A}$ such that for any $\varphi \in \mathcal{L}_{\mathrm{EL}}^{\kappa}$, variable x, and assignment s,

$$\mathcal{A}, w \vDash \varphi \text{ iff } \mathfrak{A}, s_{[x:=w]} \vDash ST_x(\varphi)$$

Under the translation, the modal language lives in the two-variable fragment of FOL. It also lives in the "guarded" fragment of FOL, since all the quantifiers have guards. The satisfiability/validity problem for formulas in both of these fragment is decidable.

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Van Benthem's Characterization Theorem gives a *semantic characterization* of the FOL formulas (with one free variable) that are equivalent to the standard translation of some modal formula: they are the FOL formulas that are invariant for bisimulation.

Monadic First-Order Logic

In the other direction, any formula φ of Monadic First-Order Logic with only x free (and not in the scope of any quantifier) can be translated into a modal formula φ' such that for any MFO structure $\mathfrak{A}$ and assignment s, when we view $\mathfrak{A}$ as a single-agent epistemic partition model $\mathcal{A}$ (with R_a the universal relation),

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But it is a long way from here to answers for full FOL.

Directions from Here

Going beyond the basic epistemic logic we have covered today:

- Quantified epistemic logic ($K_a \exists x Spy(x)$ vs. $\exists x K_a Spy(x)$);
- "Softer" belief revision instead of "hard" knowledge update;
- Modeling private communication, eavesdropping, etc.;
- Epistemic logic and probability (qualitative and quantitative);
- Philosophical applications of EL, mentioned at the beginning;
- and more...

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Two sources for learning about recents research directions are:

- *Advances in Modal Logic*, biennial conference and book series.
- *European Summer School for Logic, Language, and Information*.

Thank you!

Sum and Product

A says to S and P: "I have picked $x, y \in \mathbb{N}$ such that $1 < x < y$ and $x + y \leq 100$. Shortly I will whisper their sum $s = x + y$ to S only, and their product $p = x \cdot y$ to P only." A acts accordingly. The following conversation between S and P then takes place:

1. P: "I don't know the numbers."
2. S: "I knew that you didn't know them."
3. P: "Now I know the numbers."
4. S: "Now I know them too."

Question: what are x and y? (There is a unique answer.)