Dynamic Measure Logic

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Abstract

This paper brings together Dana Scott's measure-based semantics for the propositional modal logic $S4$, and recent work in Dynamic Topological Logic. In a series of recent talks, Scott showed that the language of $S4$ can be interpreted in the Lebesgue measure algebra, $\mathcal{M}$, or algebra of Borel subsets of the real interval, [0, 1], modulo sets of measure zero. Conjunctions, disjunctions and negations are interpreted via the Boolean structure of the algebra, and we add an interior operator on $\mathcal{M}$ that interprets the $\Box$-modality. In this paper we show how to extend this measure-based semantics to the bimodal logic of $S4C$. $S4C$ is a dynamic topological logic that is interpreted in 'dynamic topological systems,' or topological spaces together with a continuous function acting on the space. We extend Scott's measure based semantics to this bimodal logic by defining a class of operators on the algebra $\mathcal{M}$, which we call O-operators and which take the place of continuous functions in the topological semantics for $S4C$. The main result of the paper is that $S4C$ is complete for the Lebesgue measure algebra. A strengthening of this result, also proved here, is that there is a *single* measure-based model in which all non-theorems of $S4C$ are refuted.

Keywords: Modal logic, S4, Completeness, Topological Semantics, Measure Algebra

1 Introduction

Kripke models for normal modal logics, consisting of a set of possible worlds together with a binary accessibility relation, are, by now, widely familiar. But long before Kripke semantics became standard, Tarski showed that the propositional modal logic $S4$ can be interpreted in topological spaces. In the topological semantics for $S4$, a topological space is fixed, and each propositional variable, p, is assigned an arbitrary subset of the space: the set of points where p is *true*. Conjunctions, disjunctions and negations are interpreted as set-theoretic intersections, unions and complements (thus, *e.g.*, $\phi \wedge \psi$ is *true* at all points in

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the intersection of the set of points where ϕ is true and the set of points where ψ is true. The $\Box$-modality of $S4$ is interpreted via the topological interior: $\Box\phi$ is true at any point in the topological interior of the set of points at which ϕ is true.

In this semantics, the logic $S4$ can be seen as describing topological spaces. Indeed, with the topological semantics it became possible to ask not just whether $S4$ is complete for the set of topological validities—formulas valid in *every* topological space—but also whether $S4$ is complete for any *given* topological space. The culmination of Tarski's work in this area was a very strong completeness result. In 1944, Tarksi and McKinsey proved that $S4$ is complete for any dense-in-itself metric space. One particularly important case was the real line, $\mathbb{R}$, and as the topological semantics received renewed interest in recent years, more streamlined proofs of Tarksi's result for this special case emerged in, *e.g.*, [1], [3], and [9]. The logic of the real line—and of Euclidean space, more generally—is just $S4$.

The real line, however, can be investigated not just from a topological point of view, but from a measure-theoretic point of view. Here, the probability measure we have in mind is the usual Lebesgue measure on the reals. In the last several years Dana Scott introduced a new *probabilistic* or *measure-based* semantics for $S4$ that is built around Lebesgue measure on the reals and is in some ways closely related to Tarski's older topological semantics.

Scott's semantics is essentially algebraic: formulas are interpreted in the Lebesgue measure algebra, or the σ-algebra of Borel subsets of the real interval [0,1], modulo sets of measure zero (henceforth, "null sets"). Let us denote this algebra by $\mathcal{M}$. Thus elements of $\mathcal{M}$ are equivalence classes of Borel sets. In Scott's semantics, each propositional variable is assigned to some element of $\mathcal{M}$. We say the *value* of the propositional variable p is that element of the algebra to which p is assigned. Conjunctions, disjunctions and negations are assigned to meets, joins and complements in the algebra, respectively. In order to interpret the $S4$ $\Box$-modality, we add to the algebra an "interior" operator (defined below), which we construct from the collection of *open* elements in the algebra, or elements that have an open representative. Unlike in the Kripke or topological semantics, there is no notion here of truth *at a point* (or at a "world"). In [5] and [8] it was shown that $S4$ is complete for the Lebesgue measure algebra.

The introduction of a measure-based semantics for $S4$ raises a host of questions that are, at this point, entirely unexplored. Among them: What about natural extensions of $S4$? Can we give a measure-based semantics not just for $S4$ but for some of its extensions that have well-known topological interpretations?

This paper focuses on a family of logics called *dynamic topological logics.* These logics were investigated over the last fifteen years, in an attempt to describe "dynamic topological systems" by means of modal logic. A dynamic topological system is a pair $\langle X, f\rangle$, where X is a topological space and f is a continuous function on X. We can think of f as moving points in X in discrete units of time. Thus in the first moment, x is mapped to $f(x)$, then to $f(f(x))$, etc. The most basic dynamic topological logic is $S4C$. In addition to the $S4$

$\Box$-modality, it has a temporal modality, which we denote by $\bigcirc$. Intuitively, we understand the formula $\bigcirc p$ as saying that at the "next moment in time," p will be true. Thus we put: $x \in V(\bigcirc p)$ iff $f(x) \in V(p)$. In [7] and [13] it was shown that $S4C$ is incomplete for the real line, $\mathbb{R}$. However, in [14] it was shown that $S4C$ is complete for Euclidean spaces of arbitrarily large finite dimension, and in [4] it was shown that $S4C$ is complete for $\mathbb{R}^2$.

The aim of this paper is to give a measure-based semantics for the logic $S4C$, along the lines of Scott's semantics for $S4$. Again, formulas will be assigned to some element of the Lebesgue measure algebra, $\mathcal{M}$. But what about the *dynamical* aspect—*i.e.*, the interpretation of the $\bigcirc$-modality? We show that there is a very natural way of interpreting the $\bigcirc$-modality via operators on the algebra $\mathcal{M}$ that take the place of continuous functions in the topological semantics. These operators can be viewed as transforming the algebra in discrete units of time. Thus one element is sent to another in the first instance, then to another in the second instance, and so on. The operators we use to interpret $S4C$ are O-operators: ones that take "open" elements in the algebra to open elements (defined below). But there are obvious extensions of this idea: for example, to interpret the logic of *homeomorphisms* on topological spaces, one need only look at *automorphisms* of the algebra $\mathcal{M}$.

Adopting a measure-based semantics for $S4C$ brings with it certain advantages. Not only do we reap the probabilistic features that come with Scott's semtantics for $S4$, but the curious *dimensional asymmetry* that appears in the topological semantics (where $S4C$ is *in*complete for $\mathbb{R}$ but complete for $\mathbb{R}^2$) disappears in the measure-based semantics. Our main result is that the logic $S4C$ is complete for the Lebesgue-measure algebra. A strengthening of this result, also proved here, is that $S4C$ is complete for a *single* model of the Lebesgue measure algebra. Due to well-known results by Oxtoby, this algebra is isomorphic to the algebra generated by Euclidean space of arbitrary dimension. In other words, $S4C$ is complete for the reduced measure algebra generated by *any* Euclidean space.

2 Topological Semantics for $S4C$

Let the language $L_{\Box,\bigcirc}$ consist of a countable set, $PV = \{p_n \mid n \in \mathbb{N}\}$, of propositional variables, and be closed under the binary connectives $\wedge, \vee, \rightarrow, \leftrightarrow$, unary operators, $\neg, \Box, \diamond$, and a unary modal operator $\bigcirc$ (thus, $L_{\Box,\bigcirc}$ is the language of propositional $S4$ enriched with a new modality, $\bigcirc$).

Definition 2.1. *A* **dynamic topological space** *is a pair $\langle X, f\rangle$, where X is a topological space and $f : X \rightarrow X$ is a continuous function on X. A* **dynamic topological model** *is a triple, $\langle X, f, V\rangle$, where X is a topological space, $f : X \rightarrow X$ is a continuous function, and $V : PV \rightarrow \mathcal{P}(X)$ is a valuation assigning to each propositional variable a subset of X. We say that $\langle X, f, V\rangle$ is a model* over *X.*

We extend V to the set of all formulas in $L_{\Box,\bigcirc}$ by means of the following *recursive clauses:*

$V(\phi \vee \psi) = V(\phi) \cup V(\psi)$
$V(\neg\phi) = X - V(\phi)$
$V(\Box\phi) = Int(V(\phi))$
$V(\bigcirc\phi) = f^{-1}(V(\phi))$

where 'Int' denotes the topological interior.

Let $N = \langle X, f, V\rangle$ be a dynamic topological model. We say that a formula ϕ is *satisfied* at a point $x \in X$ if $x \in V(\phi)$, and we write $N, x \models \phi$. We say ϕ is true in N ($N \models \phi$) if $N, x \models \phi$ for each $x \in X$. We say ϕ is *valid* in X ($\models_X \phi$), if for any model N over X, we have $N \models \phi$. Finally, we say ϕ is *topologically valid* if it is valid in every topological space.

Definition 2.2. *The logic S4C in the language* $L_{\Box,\bigcirc}$ *is given by the following axioms:*

– *the classical tautologies,*

– *S4 axioms for* $\Box$.

(A1) $\bigcirc(\phi \vee \psi) \leftrightarrow (\bigcirc\phi \vee \bigcirc\psi)$,

(A2) $(\bigcirc\neg\phi) \leftrightarrow (\neg \bigcirc \phi)$,

(A3) $\bigcirc\Box\phi \rightarrow \Box \bigcirc \phi$ *(the axiom of continuity)*

and the rules of modus ponens and necessitation for both $\Box$ *and* $\bigcirc$. *Following [7], we use S4C both for this axiomatization and for the set of all formulas derivable from the axioms by the inference rules.*

We close this section by listing the known completeness results for *S4C* in the topological semantics.

Theorem 2.3. *(Completeness) For any formula* $\phi \in L_{\Box,\bigcirc}$*, the following are equivalent:*
(i) *S4C* $\vdash \phi$*;*
(ii) ϕ *is topologically valid;*
(iii) ϕ *is true in any finite topological space;*
(iv) ϕ *is valid in* $\mathbb{R}^n$ *for* $n \geq 2$.

Proof. The equivalence of (i)-(iii) was proved by Artemov in [2]. The equivalence of (i) and (iv) was proved by Duque in [4]. This was a strengthening of a result proved by Slavnov in [14]. $\Box$

Theorem 2.4. *(Incompleteness for* $\mathbb{R}$*) There exists* $\phi \in L_{\Box,\bigcirc}$ *such that* ϕ *is valid in* $\mathbb{R}$*, but* ϕ *is not topologically valid.*

Proof. See [7] and [13]. $\Box$

3 Kripke Semantics for $S4C$

In this section we show that the logic $S4C$ can also be interpreted in the more familiar setting of Kripke frames. It is well known that the logic $S4$ (which does not include the 'temporal' modality, $\bigcirc$) is interpreted in transitive, reflexive Kripke frames, and that such frames *just are* topological spaces of a certain kind. It follows that the Kripke semantics for $S4$ is just a special case of the topological semantics for $S4$. In this section, we show that the logic $S4C$ can be interpreted in transitive, reflexive Kripke frames with some additional 'dynamic' structure, and, again, that Kripke semantics for $S4C$ is a special case of the more general topological semantics for $S4C$. Henceforth, we assume that Kripke frames are both transitive and reflexive.

Definition 3.1. *A* **dynamic Kripke frame** *is a triple $\langle W, R, G\rangle$ where W is a set, R is a reflexive, transitive relation on W, and $G : W \to W$ is a function that is R-monotone in the following sense: for any $u, v \in W$, if uRv, then $G(u)\, R\, G(v)$.*

Definition 3.2. *A* **dynamic Kripke model** *is a pair $\langle F, V\rangle$ where $F = \langle W, R, G\rangle$ is a dynamic Kripke frame and $V : PV \to \mathcal{P}(W)$ is a valuation assigning to each propositional variable an arbitrary subset of W. We extend V to the set of all formulas in $L_{\Box,\bigcirc}$ by the following recursive clauses:*

$V(\phi \vee \psi) = V(\phi) \cup V(\psi)$
$V(\neg\phi) = W - V(\phi)$
$V(\bigcirc\phi) = G^{-1}(V(\phi))$.
$V(\Box\phi) = \{w \in W \mid v \in V(\phi) \text{ for all } v \in W \text{ such that } wRv\}$

Given a dynamic Kripke frame $K = \langle W, R, G\rangle$, we can impose a topology on W via the accessibility relation R. We define the *open* subsets of W as those subsets that are upward closed under R:

(*) $O \subseteq W$ is open iff $x \in O$ and xRy implies $y \in O$

Recall that an *Alexandroff topology* is a topological space in which arbitrary intersections of open sets are open. The reader can verify that the collection of open subsets of W includes the entire space, the empty set, and is closed under arbitrary intersections and unions. Hence, viewing $\langle W, R\rangle$ as a topological space, the space is Alexandroff.

Going in the other direction, if X is an Alexandroff topology, we can define a relation R on X by:

(@) xRy iff x is a point of closure of $\{y\}$

(Equivalently, y belongs to every open set containing x.) Clearly R is reflexive. To see that R is transitive, suppose that xRy and yRz. Let O be an open set containing x. Then since x is a point of closure for $\{y\}$, $y \in O$. But since y is a point of closure for $\{z\}$, $z \in O$. So x is a point of closure for $\{z\}$ and xRz. So far, we have shown that *static* Kripke frames, $\langle W, R\rangle$ correspond to Alexandroff topologies. But what about the dynamical aspect? Here we invite the reader to verify that R-monotonicity of the function G is equivalent to continuity of G in the topological setting. It follows that *dynamic Kripke frames are just dynamic Alexandroff topologies.*

In view of the fact that every finite topology is Alexandroff (if X is finite, then there are only finitely many open subsets of X), we have shown that finite topologies are just finite Kripke frames. This result, together with Theorem 2.3 (*iii*), gives the following completeness theorem for Kripke semantics:

Lemma 3.3. *For any formula $\phi \in L_{\Box,\bigcirc}$, the following are equivalent:*
(i) S4C $\vdash \phi$;
(ii) ϕ is true in any finite Kripke frame (= finite topological space).

In what follows, it will be useful to consider not just arbitrary finite Kripke frames, but frames that carry some additional structure. The notion we are after is that of a *stratified dynamic Kripke frame*, introduced by Slavnov in [14]. We recall his definitions below.

Definition 3.4. *Let $K = \langle W, R, G\rangle$ be a dynamic Kripke frame. A* **cone** *in K is any set $U_v = \{w \in W \mid vRw\}$ for some $v \in W$. We say that v is a* **root** *of U_v.*

Note in particular that any cone, U_v, in K is an open subset of W—indeed, the smallest open subset containing v.

Definition 3.5. *Let $K = \langle U, R, G\rangle$ be a finite dynamic Kripke frame. We say that K is* **stratified** *if there is a sequence $\langle U_1, \ldots, U_n\rangle$ of pairwise disjoint cones in K with roots $u_1, \ldots, u_n$ respectively, such that $U = \bigcup_k U_k$; $G(u_k) = u_{k+1}$ for $k < n$, and G is injective. We say the stratified Kripke frame has* depth n *and (with slight abuse of notation) we call u_1 the* **root** *of the stratified frame.*

Note that it follows from R-monotonicity of G that $G(U_k) \subseteq U_{k+1}$, for $k < n$.

Definition 3.6. *Define the function CD ("circle depth") on the set of all formulas in $L_{\Box,\bigcirc}$ inductively, as follows.*

$CD(p) = 0$ *for any propositional variable p;*
$CD(\phi \vee \psi) = max\,\{CD(\phi), CD(\psi)\}$;
$CD(\neg\phi) = CD(\phi)$;
$CD(\Box\phi) = CD(\phi)$;
$CD(\bigcirc\phi) = 1 + CD(\phi)$.

We also refer to $CD(\phi)$ as the $\bigcirc$-depth of ϕ.

Lemma 3.7. *Suppose the formula ϕ is not a theorem of S4C, and $CD(\phi) = n$. Then there is a stratified finite dynamic Kripke frame K with depth $n + 1$ such that ϕ is refuted at the root of K.*

Proof. The proof is by Lemma 3.3 and by a method of 'disjointizing' finite Kripke frames. For the details, see [14]. □

4 Algebraic Semantics for $S4C$

We saw that the topological semantics for $S4C$ is a generalization of the Kripke semantics. Can we generalize further? Just as classical propositional logic is interpreted in Boolean algebras, we would like to interpret modal logics algebraically. Tarski and McKinsey showed that this can be done for the logic $S4$, interpreting the $\Box$-modality as an interior operator on a Boolean algebra. In this section we show that the same can be done for the logic $S4C$, interpreting the $\bigcirc$-modality via O-operators on a Boolean algebra.

We will denote the top and bottom elements of a Boolean algebra by 1 and 0, respectively.

Definition 4.1. *A* **topological Boolean algebra** *is a Boolean algebra, $\mathcal{A}$, together with an* **interior operator** *I on $\mathcal{A}$ that satisfies:*

(I_1) $I1 = 1$;
(I_2) $Ia \leq a$;
(I_3) $IIa = Ia$;
(I_4) $I(a \wedge b) = Ia \wedge Ib$.

Example 4.2. *The set of all subsets $\mathcal{P}(X)$ of a topological space X with set-theoretic meets, joins and complements and where the operator I is just the topological interior operator (for $A \subseteq X$, $I(A) = Int(A)$) is a topological Boolean algebra. More generally, any collection of subsets of X that is closed under finite intersections, unions, complements and topological interiors is a topological Boolean algebra. We call any such algebra a* **topological field of sets**.

Suppose A is a topological Boolean algebra with interior operator I. We define the *open* elements in A as those elements for which

$$Ia = a \tag{1}$$

Definition 4.3. *Let A_1 and A_2 be topological Boolean algebras. We say $h : A_1 \to A_2$ is a* **Boolean homomorphism** *if h preserves Boolean operations. We say h is a* **Boolean embedding** *if h is an injective Boolean homomorphism. We say h is a* **homomorphism** *if h preserves Boolean operations and the interior operator. We say h is an* **embedding** *if h is an injective homomorphism. Finally, we say A_1 and A_2 are* **isomorphic** *if there is an embedding from A_1 onto A_2.*

Definition 4.4. *Let A_1 and A_2 be topological Boolean algebras, and let $h : A_1 \to A_2$. We say h is an* **O-map** *if*

(i) h is a Boolean homomorphism

(ii) For any c open in A_1, $h(c)$ is open in A_2.

An **O-operator** *is an O-map from a topological Boolean algebra to itself.*

Lemma 4.5. *Let A_1 and A_2 be topological Boolean algebras, with interior operators I_1 and I_2 respectively. Suppose that $h : A_1 \to A_2$ is a Boolean homomorphism. Then h is an O-map iff for every $a \in A_1$,*

$$h(I_1 a) \leq I_2(h(a)) \tag{2}$$

Proof. We let $\mathcal{G}_1$ and $\mathcal{G}_2$ denote the collection of open elements in A_1 and A_2 respectively. ($\Rightarrow$) Suppose h is an O-map. Then $h(I_1 a) \in \mathcal{G}_2$ by Definition 4.4 (*ii*). Also, $I_1 a \leq a$, so $h(I_1 a) \leq h(a)$ (h is a Boolean homomorphism, hence preserves order). Taking interiors on both sides, we have $h(I_1 a) = I_2(h(I_1 a)) \leq I_2(ha)$. ($\Leftarrow$) Suppose that for every $a \in A_1$, $h(I_1 a) \leq I_2(h(a))$. Let $c \in \mathcal{G}_1$. Then $c = I_1 c$, so $h(c) = h(I_1 c) \leq I_2(h(c))$. But also, $I_2(h(c)) \leq h(c)$. So $h(c) = I_2(h(c))$ and $h(c) \in \mathcal{G}_2$. $\square$

We are now in a position to state the algebraic semantics for the language $L_{\Box,\bigcirc}$.

Definition 4.6. *A* **dynamic algebra** *is a pair $\langle A, h \rangle$, where A is a topological Boolean algebra and h is an O-operator on A. A* **dynamic algebraic model** *is an ordered triple, $\langle A, h, V \rangle$, where A is a topological Boolean algebra, h is an O-operator on A, and $V : PV \to A$ is a valuation, assigning to each propositional variable $p \in PV$ an element of A. We say $\langle A, h, V \rangle$ is a model over A. We can extend V to the set of all formulas in $L_{\Box,\bigcirc}$ by the following recursive clauses:*

$$\begin{aligned} V(\phi \vee \psi) &= V(\phi) \vee V(\psi) \\ V(\neg\phi) &= -V(\phi) \\ V(\Box\phi) &= IV(\phi) \\ V(\bigcirc\phi) &= hV(\phi) \end{aligned}$$

(The remaining binary connectives, $\to$ and $\leftrightarrow$, and unary operator, $\Diamond$, are defined in terms of the above in the usual way.)

We define standard validity relations. Let $N = \langle A, h, V \rangle$ be a dynamic algebraic model. We say ϕ is *true* in N ($N \models \phi$) iff $V(\phi) = 1$. Otherwise, we say ϕ is *refuted* in N. We say ϕ is *valid* in A ($\models_A \phi$) if for any algebraic model N over A, $N \models \phi$. Finally, we let $DML_A = \{\phi \mid\ \models_A \phi\}$ (*i.e.*, the set of validities in A). In our terminology, soundness of $S4C$ for A is the claim: $S4C \subseteq DML_A$. Completeness of $S4C$ for A is the claim: $DML_A \subseteq S4C$.

Proposition 4.7. *(Soundness) Let A be a topological Boolean algebra. Then $S4C \subseteq DML_A$.*

Proof. We have to show that the $S4C$ axioms are valid in A and that the rules of inference preserve truth. To see that (A1) is valid, note that:

$$\begin{aligned} V(\bigcirc(\phi \vee \psi)) &= h(V(\phi) \vee V(\psi)) \\ &= h(V(\phi)) \vee h(V(\psi)) \quad (h \text{ a Boolean homomorphism}) \\ &= V(\bigcirc\phi \vee \bigcirc\psi) \end{aligned}$$

Thus $V(\bigcirc(\phi \vee \psi) \leftrightarrow (\bigcirc\phi \vee \bigcirc\psi)) = 1$. Validity of (A2) is proved similarly. For (A3), note that:

$$\begin{aligned} V(\bigcirc\Box\phi) &= h(IV(\phi)) \\ &\leq Ih(V(\phi)) \quad \text{(by Lemma 4.5)} \\ &= V(\Box \bigcirc \phi) \end{aligned}$$

So $V(\bigcirc\Box\phi) \leq V(\Box \bigcirc \phi)$ and $V(\bigcirc\Box\phi \to \Box \bigcirc \phi) = 1$. This takes care of the special $\bigcirc$-modality axioms. The remaining axioms are valid by soundness of $S4$ for any topological Boolean algebra—see *e.g.*, [11]. To see that necessitation for $\bigcirc$ preserves validity, suppose that ϕ is valid in A (*i.e.*, for every algebraic model $N = \langle A, h, V\rangle$, we have $V(\phi) = 1$). Then $V(\bigcirc\phi) = h(V(\phi)) = h(1) = 1$, and $\bigcirc\phi$ is valid in A. □

5 Reduced Measure Algebras

We would like to interpret $S4C$ not just in arbitrary topological Boolean algebras, but in algebras carrying a probability measure—or 'measure algebras.' In this section we show how to construct such algebras from separable metric spaces together with a σ-finite Borel measure (defined below).

Definition 5.1. *Let A be a Boolean σ-algebra, and let μ be a non-negative function on A. We say μ is a* **measure** *on A if for any countable collection $\{a_n\}$ of disjoint elements in A, $\mu(\bigvee_n a_n) = \sum_n \mu(a_n)$.*

If μ is a measure on A, we say μ is ***positive*** if 0 is the only element at which μ takes the value 0. We say μ is ***σ-finite*** if 1 is the countable join of elements in A with finite measure.[1] Finally, we say μ is ***normalized*** if $\mu(1) = 1$.

Definition 5.2. *A* **measure algebra** *is a Boolean σ-algebra A together with a positive, σ-finite measure μ on A.*

Lemma 5.3. *Let A be a Boolean σ-algebra and let μ be a σ-finite measure on A. Then there is a normalized measure ν on A such that for all $a \in A$, $\mu(a) = 0$ iff $\nu(a) = 0$.*

Proof. Since μ is σ-finite, there exists a countable collection $\{s_n \mid n \geq 1\} \subseteq A$ such that $\bigvee_{n\geq 1} s_n = 1$ and $\mu(s_n) < \infty$ for each $n \geq 1$. WLOG we can assume the s_n's are pairwise disjoint (*i.e.*, $s_n \wedge s_m = 0$ for $m \neq n$). For any $a \in A$, let

$$\nu(a) = \sum_{n\geq 1} 2^{-n} \frac{\mu(a \wedge s_n)}{\mu(s_n)}$$

The reader can verify that ν has the desired properties. □

[1] *I.e.*, there is a countable collection of elements A_n in A such that $\bigvee_n A_n = 1$ and $\mu(A_n) < \infty$ for each $n \in \mathbb{N}$.

In what follows, we show how to construct measure algebras from a topological space, X, together with a Borel measure on X. The relevant definition is given below.

Definition 5.4. *Let X be a topological space. We say that μ is a* **Borel measure** *on X if μ is a measure defined on the σ-algebra of Borel subsets of X.*[2]

Let X be a topological space, and let μ be a σ-finte Borel measure on X. We let $Borel(X)$ denote the collection of Borel subsets of X and let $Null_\mu$ denote the collection of measure-zero Borel sets in X. Then $Borel(X)$ is a Boolean σ-algebra, and $Null_\mu$ is a σ-ideal in $Borel(X)$. We form the quotient algebra

$$\mathcal{M}_X^\mu = Borel(X)/Null_\mu$$

(Equivalently, we can define the equivalence relation $\sim$ on Borel sets in X by $A \sim B$ iff $\mu(A \triangle B) = 0$, where $\triangle$ denotes symmetric difference. Then $\mathcal{M}_X^\mu$ is the algebra of equivalence classes under $\sim$.) Boolean operations in $\mathcal{M}_X^\mu$ are defined in the usual way in terms of underlying sets:

$$\begin{aligned} |A| \vee |B| &= |A \cup B| \\ |A| \wedge |B| &= |A \cap B| \\ -|A| &= |X - A| \end{aligned}$$

Lemma 5.5. *There is a unique measure ν on $\mathcal{M}_X^\mu$ such that $\nu|A| = \mu(A)$ for all A in $Borel(X)$. Moreover, the measure ν is σ-finite and positive.*

Proof. See [6], pg. 79. □

It follows from Lemma 5.5 that $\mathcal{M}_X^\mu$ is a measure algebra. We follow Halmos [6] in referring to any algebra of the form $\mathcal{M}_X^\mu$ as a ***reduced measure algebra***.[3]

Lemma 5.6. *Let X be a topological space and let μ be a σ-finite Borel measure on X. Then for any $|A|, |B| \in \mathcal{M}_X^\mu$, $|A| \leq |B|$ iff $A \subseteq B \cup N$ for some $N \in Null_\mu$.*

Proof. ($\Rightarrow$) If $|A| \leq |B|$, then $|A| \wedge |B| = |A|$, or equivalently $|A \cap B| = |A|$. This means that $(A \cap B) \triangle A \in Null_\mu$, so $A - B \in Null_\mu$. But $A \subseteq B \cup (A - B)$. ($\Leftarrow$) Suppose $A \subseteq B \cup N$ for some $N \in \mathrm{Null}_\mu$. Then $A \cap (B \cup N) = A$, and $|A| \wedge |B \cup N| = |A|$. But $|B \cup N| = |B|$, so $|A| \wedge |B| = |A|$, and $|A| \leq |B|$. □

For the remainder of this section, let X be a separable metric space, and let μ be a σ-finite Borel measure on X. Where the intended measure is obvious, we will drop superscripts, writing $\mathcal{M}_X$ for $\mathcal{M}_X^\mu$.

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[2] *I.e.*, on the smallest σ-algebra containing all open subsets of X.

[3] In fact, Halmos allows as 'measure algebras' only algebras with a normalized measure. We relax this constraint here, in order to allow for the 'reduced measure algebra' generated by the entire real line together with the usual Lebesgue measure. This algebra is, of course, isomorphic to $\mathcal{M}_X^\mu$, where X is the real interval [0, 1], and μ is the usual Lebesgue measure on X. This amendment was suggested by the anonymous referee.

So far we have seen only that $\mathcal{M}_X^\mu$ is a Boolean algebra. In order to interpret the $\Box$-modality of $S4C$ in $\mathcal{M}_X^\mu$, we need to construct an interior operator on this algebra (thus transforming $\mathcal{M}_X^\mu$ into a topological Boolean algebra). We do this via the topological structure of the underlying space, X. Let us say that an element $a \in \mathcal{M}_X^\mu$ is *open* if $a = |U|$ for some open set $U \subseteq X$. We denote the collection of open elements in $\mathcal{M}_X^\mu$ by $\mathcal{G}_X^\mu$ (or, dropping superscripts, $\mathcal{G}_X$).

Proposition 5.7. *$\mathcal{G}_X^\mu$ is closed under (i) finite meets and (ii) arbitrary joins.*

Proof. (i) This follows from the fact that open sets in X are closed under finite intersections. (ii) Let $\{a_i \,|\, i \in I\}$ be a collection of elements in G_X^μ. We need to show that $sup\,\{a_i \,|\, i \in I\}$ exists and is equal to some element in $\mathcal{G}_X^\mu$. Since X is separable, there exists a countable dense set D in X. Let $\mathcal{B}$ be the collection of open balls in X centered at points in D with rational radius. Then any open set in X can be written as a union of elements in $\mathcal{B}$. Let S be the collection of elements $B \in \mathcal{B}$ such that $|B| \leq a_i$ for some $i \in I$. We claim that

$$sup\,\{a_i \,|\, i \in I\} = |\bigcup S|$$

First, we need to show that $|\bigcup S|$ is an upper bound on $\{a_i \,|\, i \in I\}$. For each $i \in I$, $a_i = |U_i|$ for some open set $U_i \subseteq X$. Since U_i is open, it can be written as a union of elements in $\mathcal{B}$. Moreover, each of these elements is a member of S (if $B \in \mathcal{B}$ and $B \subseteq U_i$, then $|B| \leq |U_i| = a_i$). So $U_i \subseteq \bigcup S$ and $a_i = |U_i| \leq |\bigcup S|$.

For the reverse inequality ($\geq$) we need to show that if m is an upper bound on $\{a_i \,|\, i \in I\}$, then $|\bigcup S| \leq m$. Let $m = |M|$. Note that S is countable (since $S \subseteq \mathcal{B}$ and $\mathcal{B}$ is countable). We can write $S = \{B_n \,|\, n \in \mathbb{N}\}$. Then for each $n \in \mathbb{N}$, there exists $i \in I$ such that $|B_n| \leq a_i \leq m$. By Lemma 5.5, $B_n \subseteq M \cup N_n$ for some $N_n \in Null_\mu$. Taking unions, $\bigcup_n B_n \subseteq M \cup \bigcup_n N_n$, and $\bigcup_n N_n \in Null_\mu$. By Lemma 5.5, $|S| = |\bigcup_n B_n| \leq m$. $\Box$

We can now define an interior operator, I_X^μ, on $\mathcal{M}_X^\mu$ via the collection of open elements, $\mathcal{G}_X^\mu$. For any $a \in \mathcal{M}_X^\mu$, let

$$I_X^\mu a = sup\,\{c \in \mathcal{G}_X^\mu \,|\, c \leq a\}$$

Lemma 5.8. *I_X^μ is an interior operator.*

Proof. For simplicity of notation, we let I denote I_X^μ and let $\mathcal{G}$ denote $\mathcal{G}_X^\mu$. Then (I_1) follows from the fact that $1 \in \mathcal{G}$. (I_2) follows from the fact that a is an upper bound on $\{c \in \mathcal{G} \,|\, c \leq a\}$. For (I_3) note that by (I_2), we have $IIa \leq Ia$. Moreover, if $c \in \mathcal{G}$ with $c \leq a$, then $c \leq Ia$ (since Ia is supremum of all such c). Thus $\bigvee\{c \in \mathcal{G} \,|\, c \leq a\} \leq \bigvee\{c \in \mathcal{G} \,|\, c \leq Ia\}$, and $Ia \leq IIa$. For (I_4) note that since $a \wedge b \leq a$, we have $I(a \wedge b) \leq Ia$. Similarly, $I(a \wedge b) \leq Ib$, so $I(a \wedge b) \leq Ia \wedge Ib$. For the reverse inequality, note that $Ia \wedge Ib \leq a$ (since $Ia \leq a$), and similarly $Ia \wedge Ib \leq b$. So $Ia \wedge Ib \leq a \wedge b$. Moreover, $Ia \wedge Ib \in \mathcal{G}$. It follows that $Ia \wedge Ib \leq I(a \wedge b)$. $\Box$

Remark 5.9. *Is the interior operator* I_X^μ *non-trivial? (That is, does there exist* $a \in \mathcal{M}_X^\mu$ *such that* $Ia \neq a$*?) This depends on the space,* X*, and the measure,* μ*. If we let* X *be the real interval,* $[0,1]$*, and let* μ *be the Lebesgue measure on Borel subsets of* X*, then the interior operator is non-trivial. For the proof, see [8]. But suppose* μ *is a non-standard measure on the real interval, [0, 1], defined by:*

$$\mu(A) = \begin{cases} 1 & \textit{if } \frac{1}{2} \in A \\ 0 & \textit{otherwise} \end{cases}$$

Then $Borel([0,1])/Null_\mu$ *is the algebra 2, and both elements of this algebra are 'open.' So* $Ia = a$ *for each element* a *in the algebra.*

Remark 5.10. *The operator* I_X^μ *does not coincide with taking topological interiors on underlying sets. More precisely, it is in general not the case that for* $A \subseteq X$*,* $I_X^\mu(|A|) = |Int\ (A)|$*, where 'Int(A)' denotes the topological interior of* A*. Let* X *be the real interval* $[0,1]$ *with the usual topology, and let* μ *be Lebesgue measure restricted to measurable subsets of* X*. Consider the set* $X - \mathbb{Q}$ *and note that* $|X - \mathbb{Q}| = |X|$ *(*$\mathbb{Q}$ *is countable, hence has measure zero). We have:* $I_X^\mu(|X - \mathbb{Q}|) = I_X^\mu(|X|) = I_X^\mu(1) = 1$*. However,* $|Int(X - \mathbb{Q})| = |\emptyset| = 0$*.*

Remark 5.11. *Note that an element* $a \in \mathcal{M}_X^\mu$ *is open just in case* $I_X^\mu a = a$*. Indeed, if* a *is open, then* $a \in \{c \in \mathcal{G}_X^\mu \,|\, c \leq a\}$*. So* $a = sup\{c \in \mathcal{G}_X^\mu \,|\, c \leq a\} = I_X^\mu a$*. Also, if* $I_X^\mu a = a$*, then* a *is the join of a collection of elements in* $\mathcal{G}_X^\mu$*, and so* $a \in \mathcal{G}_X^\mu$*. This shows that the definition of 'open' elements given above fits with the definition in (1).*

In what follows, it will sometimes be convenient to express the interior operator I_X^μ in terms of underlying open sets, as in the following Lemma:

Lemma 5.12. *Let* $A \subseteq X$*. Then* $I_X^\mu(|A|) = |\bigcup\{O \textit{ open} \,|\, |O| \leq |A|\}|$

Proof. By definition of I_X^μ, $I_X^\mu(|A|) = sup\{c \in \mathcal{G}_X^\mu \,|\, c \leq |A|\}$. Let $\mathcal{B}$ and D be as in the proof of Proposition 5.7, and let S be the collection of elements $B \in \mathcal{B}$ such that $|B| \leq |A|$. Then by the proof of Proposition 5.7, $I_X^\mu(|A|) = |\bigcup S|$. But now $\bigcup S = \bigcup\{O \text{ open} \,|\, |O| \leq |A|\}$. (This follows from the fact that any open set $O \subseteq X$ can be written as a union of elements in $\mathcal{B}$.) Thus, $I_X^\mu(|A|) = |\bigcup S| = |\bigcup\{O \text{ open} \,|\, |O| \leq |A|\}|$. □

We have shown that $\mathcal{M}_X^\mu$ together with the operator I_X^μ is a topological Boolean algebra. Of course, for purposes of our semantics, we are interested in O-operators on $\mathcal{M}_X^\mu$. How do such maps arise? Unsurprisingly, a rich source of examples comes from continuous functions on the underlying topological space X. Let us spell this out more carefully.

Definition 5.13. *Let* X *and* Y *be topological spaces and let* μ *and* ν *be Borel measures on* X *and* Y *respectively. We say* $f : X \to Y$ *is* **measure-zero preserving** *(MZP) if for any* $A \subseteq Y$*,* $\nu(A) = 0$ *implies* $\mu(f^{-1}(A)) = 0$*.*

Lemma 5.14. *Let X and Y be separable metric spaces, and let μ and ν be σ-finite Borel measures on X and Y respectively. Suppose B is a Borel subset of X with $\mu(B) = \mu(X)$, and $f : B \to Y$ is measure-zero preserving and continuous. Define $h_f^{|\cdot|} : \mathcal{M}_Y^\nu \to \mathcal{M}_X^\mu$ by*

$$h_f^{|\cdot|}(|A|) = |f^{-1}(A)|$$

Then $h_f^{|\cdot|}$ is an O-map. In particular, if $X = Y$, then $h_f^{|\cdot|}$ is an O-operator.

Proof. First, we must show that $h_f^{|\cdot|}$ is well-defined.[4] Indeed, if $|A| = |B|$, then $\nu(A \triangle B) = 0$. And since f is MZP, $\mu\left(f^{-1}(A) \triangle f^{-1}(B)\right) = \mu\left(f^{-1}(A \triangle B)\right) = 0$. So $f^{-1}(A) \sim f^{-1}(B)$. This shows that $h_f^{|\cdot|}|A|$ is independent of the choice of representative, A. Furthermore, it is clear that $h_f^{|\cdot|}$ is a Boolean homomorphism. To see that it is an O-map, we need only show that if $c \in \mathcal{G}_Y^\nu$, $h_f^{|\cdot|}(c) \in \mathcal{G}_X^\mu$. But if $c \in \mathcal{G}_Y^\nu$ then $c = |U|$ for some open set $U \subseteq Y$. By continuity of f, $f^{-1}(U)$ is open in B. So $f^{-1}(U) = O \cap B$ for some O open in X. So $h_f^{|\cdot|}(c) = |f^{-1}(U)| = |O| \in \mathcal{G}_X^\mu$. □

By the results of the previous section, we can now interpret the language of *S4C* in reduced measure algebras. In particular, we say an algebraic model $\langle A, h, V\rangle$ is a ***dynamic measure model*** if $A = \mathcal{M}_X^\mu$ for some separable metric space X and a σ-finite Borel measure μ on X.

We are particularly interested in the reduced measure algebra generated by the real interval, [0, 1], together with the usual Lebesgue measure.

Definition 5.15. *(Lebesgue Measure Algebra) Let I be the real interval $[0,1]$ and let λ denote Lebesgue measure restricted to the Borel subsets of I. The* **Lebesgue measure algebra** *is the algebra $\mathcal{M}_I^\lambda$.*

Because of it's central importance, we denote the Lebesgue measure algebra without subscripts or superscripts, by $\mathcal{M}$. Furthermore, we denote the collection of open elements in $\mathcal{M}$ by $\mathcal{G}$ and the interior operator on $\mathcal{M}$ by I.

As in Definition 4.6, we let $DML_{\mathcal{M}} = \{\phi \mid\ \models_{\mathcal{M}} \phi\}$ (*i.e.*, the set of validities in $\mathcal{M}$). In our terminology, soundness of *S4C* for $\mathcal{M}$ is the claim: $S4C \subseteq DML_{\mathcal{M}}$. Completeness of *S4C* for $\mathcal{M}$ is the claim: $DML_{\mathcal{M}} \subseteq S4C$.

Proposition 5.16. *(Soundness) $S4C \subseteq DML_{\mathcal{M}}$.*

Proof. Immediate from Proposition 4.7. □

Remark 5.17. *The algebra $\mathcal{M}$ is isomorphic to the algebra $Leb([0,1])/Null_\mu$ where $Leb([0,1])$ is the σ-algebra of Lebesgue-measureable subsets of the real interval $[0,1]$, and $Null_\mu$ is the σ-ideal of Lebesgue measure-zero sets. This follows from the fact that every Lebesgue-measureable set in $[0,1]$ differs from some Borel set by a set of measure zero.*

[4] Note that by continuity of f, $f^{-1}(A)$ is a Borel set in B, hence also a Borel set in X.

6 Isomorphism between Reduced Measure Algebras

In this section we use a well-known result of Oxtoby's to show that any reduced measure algebra generated by a topologically complete, separable metric space with a σ-finite, nonatomic Borel measure is isomorphic to $\mathcal{M}$. By Oxtoby's result, we can think of $\mathcal{M}$ as *the* *canonical* separable measure algebra.

In the remainder of this section, let $\mathcal{J}$ denote the space $[0,1] - \mathbb{Q}$ (with the usual metric topology), and let δ denote Lebesgue measure restricted to the Borel subsets of $\mathcal{J}$.

Definition 6.1. *A topological space X is* **topologically complete** *if X is homeomorphic to a complete metric space.*

Definition 6.2. *Let X be a topological space. A Borel measure μ on X is* **nonatomic** *if $\mu(\{x\}) = 0$ for each $x \in X$.*

Theorem 6.3. *(Oxtoby, 1970) Let X be a topologically complete, separable metric space, and let μ be a normalized, nonatomic Borel measure on X. Then there exists a Borel set $B \subseteq X$ and a function $f : B \to \mathcal{J}$ such that $\mu(X-B) = 0$ and f is a measure-preserving homeomorphism (where the measure on $\mathcal{J}$ is δ).*

Proof. See [10]. □

Lemma 6.4. [5] *Suppose X and Y are separable metric spaces, and μ and ν are normalized Borel measures on X and Y respectively. If $f : X \to Y$ is a measure preserving homoemorphism, then $\mathcal{M}_X^\mu$ is isomorphic to $\mathcal{M}_Y^\nu$.*

Proof. For simplicity of notation, we drop superscripts, writing simply $\mathcal{M}_X$, $\mathcal{G}_X$, and I_X, etc. Let $h_f^{|\cdot|} : \mathcal{M}_Y \to \mathcal{M}_X$ be defined by $h_f^{|\cdot|}(|A|) = |f^{-1}(A)|$. This function is well-defined because f is MZP and continuous. (The first property ensures that $h_f^{|\cdot|}(|A|)$ is independent of representative A; the second ensures that $f^{-1}(A)$ is Borel.) Clearly $h_f^{|\cdot|}$ is a Boolean homomorphism. We can define the mapping $h_{f^{-1}}^{|\cdot|} : \mathcal{M}_X \to \mathcal{M}_Y$ by $h_{f^{-1}}^{|\cdot|}(|A|) = |f(A)|$. Then $h_f^{|\cdot|}$ and $h_{f^{-1}}^{|\cdot|}$ are inverses, so $h_f^{|\cdot|}$ is bijective. We need to show that $h_f^{|\cdot|}$ preserves interiors—*i.e.*, $h_f^{|\cdot|}(I_Y a) = I_X h_f^{|\cdot|}(a)$. The inequality ($\leq$) follows from the fact that $h_f^{|\cdot|}$ is an O-map (see Lemma 5.14). For the reverse inequality, we need to see that $h_f^{|\cdot|}(I_Y a)$ is an upper bound on $\{c \in \mathcal{G}_X \mid c \leq h_f^{|\cdot|}(a)\}$. If $c \in \mathcal{G}_X$, then $h_{f^{-1}}^{|\cdot|}(c) \in \mathcal{G}_Y$ and if $c \leq h_f^{|\cdot|}(a)$, then $h_{f^{-1}}^{|\cdot|}(c) \leq h_{f^{-1}}^{|\cdot|}(h_f^{|\cdot|}(a)) = a$. Thus $h_{f^{-1}}^{|\cdot|}(c) \leq I_Y a$, and $c = h_f^{|\cdot|}(h_{f^{-1}}^{|\cdot|}(c)) \leq h_f^{|\cdot|}(I_Y a)$. □

[5] We can relax the conditions of the lemma, so that instead of requiring that f is measure-preserving, we require only that $\nu(f(S)) = 0$ iff $\mu(A) = 0$. In fact, we can further relax these conditions so that $f : B \to C$, where $B \subseteq X$, $C \subseteq Y$, $\mu(B \triangle X) = 0$, and $\nu(C \triangle Y) = 0$. We prove the lemma as stated because only this weaker claim is needed for the proof of Corollary 6.5.

Corollary 6.5. *Let X be a separable metric space, and let μ be a nonatomic σ-finite Borel measure on X with $\mu(X) > 0$. Then,*

$$\mathcal{M}_X^\mu \cong \mathcal{M}$$

Proof. By Lemma 5.3, we can assume that μ is normalized.[6] Let X_{comp} be the completion of the metric space X. Clearly X_{comp} is separable. We can extend the Borel measure μ on X to a Borel measure μ^* on X_{comp} by letting $\mu^*(S) = \mu(S \cap X)$ for any Borel set S in X_{comp}. The reader can convince himself that μ^* is a normalized, nonatomic, σ-finite Borel measure on X_{comp}, and that $\mathcal{M}_{X_{comp}}^{\mu^*} \cong \mathcal{M}_X^\mu$. By Theorem 6.3, there exists a set $B \subseteq X_{comp}$ and a function $f : B \to \mathcal{J}$ such that $\mu(B) = 1$ and f is a measure-preserving homeomorphism. By Lemma 6.4, $\mathcal{M}_\mathcal{J} \cong \mathcal{M}_B$. We have:

$$\mathcal{M} \cong \mathcal{M}_\mathcal{J} \cong \mathcal{M}_B \cong M_{X_{comp}}^{\mu^*} \cong \mathcal{M}_X^\mu$$

□

7 Invariance Maps

At this point, we have at our disposal two key results: completeness of $S4C$ for finite stratified Kripke frames, and the isomorphism between $\mathcal{M}_X^\mu$ and $\mathcal{M}$ for any separable metric space X and σ-finite, nonatomic Borel measure μ. Our aim in what follows will be to transfer completeness from finite stratified Kripke frames to the Lebesgue measure algebra, $\mathcal{M}$. But how to do this?

We can view any topological space as a topological Boolean algebra—indeed, as the topological field of all subsets of the space (see Example 4.2). Viewing the finite stratified Kripke frames in this way, what we need is 'truth-preserving' maps between the algebras generated by Kripke frames and $\mathcal{M}_X^\mu$, for appropriately chosen X and μ. The key notion here is that of a "dynamic embedding" (defined below) of one dynamic algebra into another. Although our specific aim is to transfer truth from Kripke algebras to reduced measure algebras, the results we present here are more general and concern truth preserving maps between arbitrary dynamic algebras.

Recall that a *dynamic algebra* is a pair $\langle A, h \rangle$, where A is a topological Boolean algebra, and h is an O-operator on A.

Definition 7.1. *Let $M_1 = \langle A_1, h_1 \rangle$ and $M_2 = \langle A_2, h_2 \rangle$ be two dynamic algebras. We say a function $h : M_1 \to M_2$ is a* **dynamic embedding** *if*

(i) h is an embedding of A_1 into A_2;

(ii) $h \circ h_1 = h_2 \circ h$.

[6] More explicitly: If μ is σ-finite, then by Lemma 5.3 there is a normalized Borel measure μ^* on X such that $\mu^*(S) = 0$ iff $\mu(S) = 0$ for each $S \subseteq X$. It follows that $\mathcal{M}_X^\mu \cong \mathcal{M}_X^{\mu^*}$ (where the isomorphism is not, in general, measure-preserving).

Lemma 7.2. *Let $M_1 = \langle A_1, h_1, V_1\rangle$ and $M_2 = \langle A_2, h_2, V_2\rangle$ be two dynamic algebraic models. Suppose that $h : \langle A_1, h_1\rangle \to \langle A_2, h_2\rangle$ is a dynamic embedding, and for every propositional variable p,*

$$V_2(p) = h \circ V_1(p)$$

Then for any $\phi \in L_{\Box,\bigcirc}$,

$$V_2(\phi) = h \circ V_1(\phi)$$

Proof. By induction on the complexity of ϕ. □

Corollary 7.3. *Let $M_1 = \langle A_1, h_1, V_1\rangle$ and $M_2 = \langle A_2, h_2, V_2\rangle$ be two dynamic algebraic models. Suppose that $h : \langle A_1, h_1\rangle \to \langle A_2, h_2\rangle$ is a dynamic embedding, and for every propositional variable p,*

$$V_2(p) = h \circ V_1(p)$$

Then for any $\phi \in L_{\Box,\bigcirc}$,

$$M_1 \models \phi \quad \textit{iff} \quad M_2 \models \phi$$

Proof. $M_2 \models \phi$ iff $V_2(\phi) = 1$
iff $h \circ V_1(\phi) = 1$ (by Lemma 7.2)
iff $V_1 = 1$ (since h is an embedding)

□

Let $\langle X, F\rangle$ be a dynamic topological space and let A_X be the topological field of all subsets of X (see Example 4.2). We define the function h_F on A_X by

$$h_F(S) = F^{-1}(S)$$

It is not difficult to see that h_F is an O-operator. We say that $\langle A_X, h_F\rangle$ is the dynamic algebra ***generated by*** (or ***corresponding to***) to the dynamic topological space $\langle X, F\rangle$.

Our goal is to embed the dynamic algebras generated by finite dynamic Kripke frames into a dynamic measure algebra, $\langle \mathcal{M}^\mu_X, h\rangle$, where X is some appropriately chosen separable metric space and μ is a nonatomic, σ-finite Borel measure on X. In view of Corollary 7.3 and completeness for finite dynamic Kripke frames, this will give us completeness for the measure semantics. The basic idea is to construct such embeddings via 'nice' maps on the underlying topological spaces. To this end, we introduce the following new definition:

Definition 7.4. *Suppose X and Y are a topological spaces, and μ is a Borel measure on X. Let $\gamma : X \to Y$. We say γ has the* **M-property** *with respect to μ if for any subset $S \subseteq Y$:*

(i) *$\gamma^{-1}(S)$ is Borel;*

(ii) *for any open set $O \subseteq X$, if $\gamma^{-1}(S) \cap O \neq \emptyset$ then $\mu(\gamma^{-1}(S) \cap O) > 0$.*

Lemma 7.5. *Suppose $\langle X, F\rangle$ is a dynamic topological space, where X is a separable metric space, F is measure-zero preserving, and let μ be a σ-finite Borel measure on X with $\mu(X) > 0$. Suppose $\langle Y, G\rangle$ is a dynamic topological space, and $\langle A_Y, h_G\rangle$ is the corresponding dynamic algebra. Let B be a subset of X with $\mu(B) = \mu(X)$, and suppose we have a map $\gamma : B \to Y$ that satisfies:*

(i) *γ is continuous, open and surjective;*

(ii) *$\gamma \circ F = G \circ \gamma$;*

(iii) *γ has the M-property with respect to μ.*

Then the map $\Phi : \langle A_Y, h_G\rangle \to \langle \mathcal{M}^{\mu}_X, h^{|\cdot|}_F\rangle$ defined by

$$\Phi(S) = |\gamma^{-1}(S)|$$

is a dynamic embedding.

Proof. By the fact that $\mathcal{M}^{\mu}_X$ is isomorphic to $\mathcal{M}^{\mu}_B$, we can view Φ as a map from $\langle A_Y, h_G\rangle$ into $\langle \mathcal{M}^{\mu}_B, h^{|\cdot|}_F\rangle$, where h^{μ}_F is viewed as an operator on $\mathcal{M}^{\mu}_B$. Note that Φ is well-defined by the fact that γ satisfies clause (i) of the M-property. We need to show that (i) Φ is an embedding of $\langle A_Y, h_G\rangle$ into $\langle \mathcal{M}^{\mu}_B, h^{|\cdot|}_F\rangle$, and (ii) $\Phi \circ h_G = h^{|\cdot|}_F \circ \Phi$.

(i) Clearly Φ is a Boolean homomorphism. We prove that Φ is injective and preserves interiors.

- (Injectivity) Suppose $\Phi(S_1) = \Phi(S_2)$ and $S_1 \neq S_2$. Then $\gamma^{-1}(S_1) \sim \gamma^{-1}(S_2)$, and $S_1 \triangle S_2 \neq \emptyset$. Let $y \in S_1 \triangle S_2$. By surjectivity of γ, we have $\gamma^{-1}(y) \neq \emptyset$. Moreover, $\mu(\gamma^{-1}(y)) > 0$ (since γ has the M-property w.r.t. μ, and the entire space B is open). So $\mu(\gamma^{-1}(S_1) \triangle \gamma^{-1}(S_2)) = \mu(\gamma^{-1}(S_1 \triangle S_2)) \geq \mu(\gamma^{-1}(y)) > 0$. And $\gamma^{-1}(S_1) \not\sim \gamma^{-1}(S_2)$. $\perp$.
- (Preservation of Interiors) For clarity, we will denote the topological interior in the spaces Y and B by Int_Y and Int_B respectively, and the interior operator on $\mathcal{M}^{\mu}_B$ by I. Let $S \subseteq Y$. It follows from continuity and openness of $\gamma : B \to Y$, that

$$\gamma^{-1}(Int_Y(S)) = Int_B(\gamma^{-1}(S))$$

  Note that,
  - $\Phi(Int_Y(S)) = |\,\gamma^{-1}(Int_Y(S))\,|$
    $= |\,Int_B(\gamma^{-1}(S))\,|$
    $= |\bigcup\{O \text{ open in } B \mid O \subseteq \gamma^{-1}(S)\}\,|$
  - $I(\Phi(S)) = I\,|\gamma^{-1}(S)|$
    $= |\bigcup\{O \text{ open in } B \mid |O| \leq |\gamma^{-1}(S)|\}\,|$ (by Lemma 5.12)

Thus it is sufficient to show that for any open set $O \subseteq B$,

$$O \subseteq \gamma^{-1}(S) \text{ iff } |O| \leq |\gamma^{-1}(S)|$$

The left-to-right direction is obvious. For the right-to-left direction, suppose (toward contradiction) that $|O| \leq |\gamma^{-1}(S)|$ but that $O \not\subseteq \gamma^{-1}(S)$. Then $O \subseteq \gamma^{-1}(S) \cup N$ for some $N \subseteq B$ with $\mu(N) = 0$. Moreover, since $O \not\subseteq \gamma^{-1}(S)$, there exists $x \in O$ such that $x \notin \gamma^{-1}(S)$. Let $y = \gamma(x)$. Then $\gamma^{-1}(y) \cap O \neq \emptyset$. Since γ has the M-property with respect to μ, it follows that $\mu(\gamma^{-1}(y) \cap O) > 0$. But $\gamma^{-1}(y) \cap O \subseteq N$ (since $\gamma^{-1}(y) \cap O \subseteq O \subseteq \gamma^{-1}(S) \cup N$, and $\gamma^{-1}(y) \cap \gamma^{-1}(S) = \emptyset$). $\perp$.

We've shown that Φ is an embedding of $\langle A_Y, h_G \rangle$ into $\langle \mathcal{M}_B^\mu, h_F^{|\cdot|} \rangle$. In view of the isomorphism between $\mathcal{M}_X^\mu$ and $\mathcal{M}_B^\mu$, we have shown that Φ is an embedding of $\langle A_Y, h_G \rangle$ into $\mathcal{M}_X^\mu$.

(*ii*) We know that $\gamma \circ F = G \circ \gamma$. Taking inverses, we have $F^{-1} \circ \gamma^{-1} = \gamma^{-1} \circ G^{-1}$. Now let $S \subseteq Y$. Then:

$$\begin{aligned} \Phi \circ h_G(S) &= |\gamma^{-1}(G^{-1}(S))| \\ &= |F^{-1}(\gamma^{-1}(S))| \\ &= h_F^{|\cdot|} \circ \Phi(S) \end{aligned}$$

□

8 Completeness of $S4C$ for the Lebesgue Measure Algebra

In this section we prove the main result of the paper: Completeness of $S4C$ for the Lebesgue measure algebra, $\mathcal{M}$. Recall that completeness is the claim that $DML_{\mathcal{M}} \subseteq S4C$. In fact, we prove the contrapositive: For any formula $\phi \in L_{\Box,\bigcirc}$, if $\phi \notin S4C$, then $\phi \notin DML_{\mathcal{M}}$. Our strategy is as follows. If ϕ is a non-theorem of $S4C$, then by Lemma 3.7, ϕ is refuted in some finite stratified Kripke frame $K = \langle W, R, G \rangle$. Viewing the frame algebraically (*i.e.*, as a topological field of sets), we must construct a dynamic embedding $\Phi : \langle A_W, h_G \rangle \to \langle \mathcal{M}, h \rangle$, where $\langle A_W, h_G \rangle$ is the dynamic Kripke algebra generated by the dynamic Kripke frame K, and h is some O-operator on $\mathcal{M}$. In view of the isomorphism between $\mathcal{M}$ and $\mathcal{M}_X^\mu$ for any separable metric space, X, and nonatomic, σ-finite Borel measure μ on X with $\mu(X) > 0$, it is enough to construct a dynamic embedding of the Kripke algebra into $\mathcal{M}_X^\mu$, for appropriately chosen X and μ.

The constructions in this section are a modification of the constructions introduced in [14], where it is proved that $S4C$ is complete for topological models in Euclidiean spaces of arbitrarily large finite dimension. The modifications we make are measure-theoretic, and are needed to accommodate the new 'probabilistic' setting. We are very much indebted to Slavnov for his pioneering work in [14].[7]

8.1 Outline of the Proof

Let us spell out the plan for the proof a little more carefully. The needed ingredients are all set out in Lemma 7.5. Our first step will be to construct the dynamic topological space $\langle X, F\rangle$, where X is a separable metric space, and F is a measure-zero preserving, continuous function on X. We must also construct a measure μ on the Borel sets of X that is nonatomic and σ-finite, such that $\mu(X) > 0$. We want to embed the Kripke algebra $\langle A_W, h_G\rangle$ into $\langle \mathcal{M}^{\mu}_{X}, h^{|\cdot|}_{F}\rangle$, and to do this, we must construct a topological map $\gamma : B \to W$, where $B \subseteq X$ and $\mu(B) = 1$, and γ satisfies the requirements of Lemma 7.5. In particular, we must ensure that (i) γ is open, continuous and surjective, (ii) $\gamma \circ F = G \circ \gamma$ and (iii) γ has the M-property with respect to μ.

In Section 8.2, we show how to construct the dynamic space $\langle X, F\rangle$, and the Borel measure μ on X. In Section 8.3, we construct the map $\gamma : X \to W$, and show that it has the desired properties.

8.2 The Topological Carrier of the Countermodel

Let

$$X_n = I^1 \sqcup \cdots \sqcup I^n$$

where I^k is the k-th dimensional unit cube and $\sqcup$ denotes disjoint union. We would like X_n to be a metric space, so we think of the cubes I^k as embedded in the space $\mathbb{R}^n$, and as lying at a certain fixed distance from one another. For simplicity of notation, we denote points in I^k by $(x_1, \ldots, x_k)$, and do not worry about how exactly these points are positioned in $\mathbb{R}^n$.

For each $k < n$, define the map $F_k : I^k \to I^{k+1}$ by $(x_1, \ldots, x_k) \mapsto (x_1, \ldots, x_k, \frac{1}{2})$. We let

$$F(x) = \begin{cases} F_k(x) & \text{if } x \in I_k, k < n \\ x & \text{if } x \in I_n \end{cases}$$

Clearly F is injective. For each $k \geq 2$ we choose a privileged "midsection" $D_k = [0,1]^{k-1} \times \{\frac{1}{2}\}$ of I_k. Thus, $f(I_k) = D_{k+1}$ for $k < n$.

The space X_n will be the carrier of our countermodels (we will choose n according to the $\bigcirc$-depth of the formula which we are refuting, as explained in the next section). We define a non-standard measure, μ, on X_n. This somewhat unusual measure will allow us to transfer countermodels on Kripke frames back to the measure algebra, $\mathcal{M}^{\mu}_{X_n}$.

[7]Where possible, we have preserved Slavnov's original notation in [14].

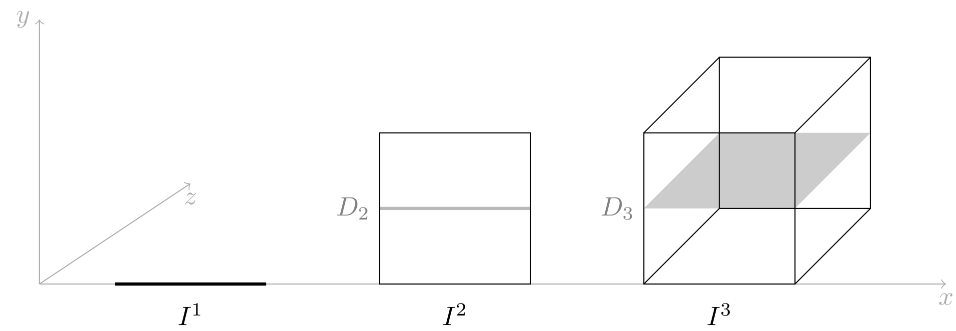

Figure 1: The space $X_3 = I^1 \sqcup I^2 \sqcup I^3$. Note that $\mu(I^1) = 1$, $\mu(I^2) = 2$, and $\mu(I^3) = 3$. The shaded regions in I^2 and I^3 denote the midsections, D_2 and D_3, respectively.

Let μ on I_1 be Lebesgue measure on $\mathbb{R}$ restricted to Borel subsets of I_1. Suppose we have defined μ on $I^1, \ldots, I^k$. For any Borel set B in I^{k+1}, let $B_1 = B \cap D_{k+1}$, and $B_2 = B \setminus D_{k+1}$. Then $B = B_1 \sqcup B_2$. We define

$$\mu(B) = \mu(F^{-1}(B_1)) + \lambda(B_2)$$

where λ is the usual Lebesgue measure in $\mathbb{R}^{k+1}$. Finally, for any Borel set $B \subseteq X_n$, we let $\mu(B) = \sum_{k=1}^{n} \mu(B \cap I^k)$

Note that $\mu(I^1) = 1$, and in general $\mu(I^{k+1}) = \mu(I^k) + 1$. Thus $\mu(X_n) = \mu(I^1 \sqcup \cdots \sqcup I^n) = \sum_1^n k = \frac{1}{2}(n^2 + n)$.

Lemma 8.1. *μ is a nonatomic, σ-finite Borel measure on X_n.*

Proof. Clearly μ is nonatomic. Moreover, since $\mu(X_n) < \infty$, μ is σ-finite. The only thing left to show is that μ is countably additive. Suppose that $\{B_m\}_{m\in\mathbb{N}}$ is a collection of pairwise disjoint subsets of X_n.

Claim 8.2. *For any $k \leq n$,*

$$\mu\,(\bigcup_m (B_m \cap I^k)) = \sum_m \mu(B_m \cap I^k)$$

(Proof of Claim: By induction on k.[8])
But now we have:

[8] The base case is by countable additivity of Lebesgue measure on the unit interval, [0, 1]. For the induction step, suppose the claim is true for $k - 1$. Then we have:

$$
\begin{aligned}
\mu(\bigcup_m B_m) &= \sum_k \mu[(\bigcup_m B_m) \cap I^k] \quad \text{(by definition of } \mu) \\
&= \sum_k \mu[\bigcup_m (B_m \cap I^k)] \\
&= \sum_k \sum_m \mu(B_m \cap I^k) \quad \text{(by Claim 8.2)} \\
&= \sum_m \sum_k \mu(B_m \cap I^k) \\
&= \sum_m \mu(B_m) \quad \text{(by definition of } \mu)
\end{aligned}
$$

□

Lemma 8.3. *X is a separable metric space and $F : X_n \to X_n$ is measure-preserving and continuous.*

Proof. The set of rational points in I^k is dense in k ($k \leq n$), so X_n is separable. Continuity of F follows from the fact that F is a translation in $\mathbb{R}^n$; F is measure-preserving by the construction of μ. □

8.3 Completeness

Assume we are given a formula $\phi \in L_{\Box,\bigcirc}$ such that ϕ is not a theorem of $S4C$ and let $n = CD(\phi) + 1$. By Lemma 3.7, there is a finite stratified, dynamic Kripke model $K = \langle W, R, G, V_1 \rangle$ of depth n such that ϕ is refuted at the root of K. In other words, there is a collection of pairwise disjoint cones $W_1, \ldots, W_n$ with roots $w_0^1, \ldots, w_0^n$ respectively, such that $W = \bigcup_{k \leq n} W_k$; G is injective; and $G(w_k) = w_{k+1}$ for each $k < n$; and $K, w_0^1 \not\models \phi$. Let the space $X = X_n = I^1 \sqcup \cdots \sqcup I^n$ and the measure μ be as defined in the previous section. We construct a map $\tilde{\gamma} : X \to W$ in a countable number of stages. To do this we will make crucial use of the notion of ϵ-nets, defined below:

Definition 8.4. *Given a metric space S and $\epsilon > 0$, a subset Ω of S is an* **ϵ-net for** *S if for any $y \in S$, there exists $x \in \Omega$ such that $d(x, y) < \epsilon$ (where d denotes the distance function in S).*

Observe that if S is compact, then for any $\epsilon > 0$ there is a *finite* ϵ-net for S.

$$
\begin{aligned}
\mu(\bigcup_m (B_m \cap I^k)) &= \mu\,[F^{-1}(\bigcup_m (B_m \cap I^k \cap D^k))] + \lambda\,[\bigcup_m (B_m \cap I^k) \setminus D^k] \quad \text{(by definition of } \mu) \\
&= \mu\,[\bigcup_m F^{-1}(B_m \cap I^k \cap D^k)] + \sum_m \lambda((B_m \cap I^k) \setminus D^k) \quad \text{(by countable additivity of } \lambda) \\
&= \sum_m \mu[F^{-1}(B_m \cap I^k \cap D^k)] + \sum_m \lambda((B_m \cap I^k) \setminus D^k) \quad \text{(by induction hypothesis)} \\
&= \sum_m \mu[F^{-1}(B_m \cap I^k \cap D^k)] + \lambda((B_m \cap I^k) \setminus D^k) \\
&= \sum_m (B_m \cap I^k) \quad \text{(by definition of } \mu)
\end{aligned}
$$

Basic Construction. Let $w^1_{\text{root}} = w^1_0$, and let $w_1, \ldots, w_{r_1}$ be the R-successors of w^1_{root}. At the first stage, we select r_1 pairwise disjoint closed cubes $T_1, \ldots, T_{r_1}$ in I^1, making sure that their total measure adds up to no more than $(\frac{1}{2})^{0+2}$—that is, $\sum_{k \leq r_1} \mu(T_k) < \frac{1}{4}$. For each x in the interior of T_k we let $\tilde{\gamma}(x) = w_k$ ($k \leq r_1$). With slight abuse of notation we put $\tilde{\gamma}(T_k) = w_k$. We refer to $T_1, \ldots, T_{r_1}$ as *terminal cubes*, and we let $I^1_1 = I^1 - \bigcup_{k=1}^{r_1} \text{Int }(T_k)$.

At any subsequent stage, we assume we are given a set I^1_i that is equal to I^1 with a finite number of open cubes removed from it. Thus I^1_i is a compact set. We find a $\frac{1}{2^i}$-net Ω_i for I^1_i and for each point $y \in \Omega_i$, we choose r_1 pairwise disjoint closed cubes, $T^y_1, \ldots, T^y_{r_1}$ in the $\frac{1}{2^i}$-neighborhood of y, putting $\tilde{\gamma}(T^y_k) = w_k$ (for $k \leq r_1$, with the same meaning as above). Again, we refer to the T_k's as terminal cubes. Since Ω_i is finite, we create only a finite number of new terminal cubes at this stage, and we make sure to do this in such a way as to remove a total measure of no more than $(\frac{1}{2})^{i+2}$. We let I^1_{i+1} be the set I^1_i minus the interiors of the new terminal cubes.

After doing this countably many times, we are left with some points in I^1 that do not belong to the interior of any terminal cube. We call such points *exceptional points* and we put $\tilde{\gamma}(x) = w^1_{\text{root}}$ for each exceptional point $x \in I^1$. This completes the definition of $\tilde{\gamma}$ on I^1.

Now assume that we have already defined $\tilde{\gamma}$ on I^j. We let $w^{j+1}_{\text{root}} = w^{j+1}_0$ and let $w_1, \ldots, w_{r_{j+1}}$ be the R-successors of w^{j+1}_{root}. We define $\tilde{\gamma}$ on I^{j+1} as follows. At first we choose r_{j+1} closed cubes $T_1, \ldots, T_{r_{j+1}}$ in I^{j+1}, putting $\tilde{\gamma}(T_k) = w_k$ (for $k \leq r_{j+1}$). In choosing $T_1, \ldots, T_{r_{j+1}}$, we make sure that these cubes are not only pairwise disjoint (as before) but also disjoint from the midsection D_{j+1}. Again, we also make sure to remove a total measure of no more than $(\frac{1}{2})^{0+2}\, \mu(I^{j+1})$. We let $I^{j+1}_1 = I^{j+1} - \bigcup_{k=1}^{r_{j+1}} \text{Int}(T_k)$.

At stage i, we assume we are given a set I^{j+1}_i equal to I^{j+1} minus the interiors of a finite number of closed cubes. Thus I^{j+1}_i is compact, and we choose a finite $\frac{1}{2^i}$-net Ω_i for I^{j+1}_i. For each $y \in \Omega_i$ we choose r_{j+1} closed terminal cubes $T_1, \ldots, T_{r_{j+1}}$ in the $\frac{1}{2^i}$-neighborhood of y. We make sure that these cubes are not only pairwise disjoint, but disjoint from the midsection D_{j+1}. Since Ω_i is finite, we add only finitely many new terminal cubes in this way. It follows that there is an ϵ-neighborhood of D_{j+1} that is disjoint from all the terminal cubes added up to this stage. Moreover, for each terminal cube T of I^j defined at the ith stage, $F(T) \subseteq D_{j+1}$, and we let T' be some closed cube in I^{j+1} containing $F(T)$ and of height at most ϵ. To ensure that the equality $\tilde{\gamma} \circ F(x) = G \circ \tilde{\gamma}(x)$ holds for all points x belonging to the interior of terminal cubes of I^j, we put:

$$\tilde{\gamma}(T') = G \circ \tilde{\gamma}(T)$$

Finally, we have added only finitely many terminal cubes at this stage, and we do so in such a way as to make sure that the total measure of these cubes is no more than $(\frac{1}{2})^{i+2}\, \mu(I^{j+1})$. We let I^{j+1}_{i+1} be the set I^{j+1}_i minus the new terminal cubes added at this stage.

We iterate this process countably many times, removing a countable number of terminal cubes from I^{j+1}. For all exceptional points x in I^{j+1} (*i.e.*, points that do not belong to the interior of any terminal cube defined at any stage) we put $\tilde{\gamma}(x) = w^{j+1}_{\mathrm{root}}$. Noting that exceptional points of I^j are pushed forward under F to exceptional points in I^{j+1}, we see that the equality $\tilde{\gamma} \circ F(x) = G \circ \tilde{\gamma}(x)$ holds for exceptional points as well.

This completes the construction of $\tilde{\gamma}$ on X. We pause now to prove two facts about the map $\tilde{\gamma}$ that will be of crucial importance in what follows.

Lemma 8.5. *Let $E(I^j)$ be the collection of all exceptional points in I^j for some $j \leq n$. Then $\mu(E(I^j)) \geq \frac{1}{2}\,\mu(I^j)$.*

Proof. At stage i of construction of $\tilde{\gamma}$ on I^j, we remove from I^j terminal cubes of total measure no more than $(\frac{1}{2})^{i+2}\,\mu(I^j)$. Thus over countably many stages we remove a total measure of no more than $\mu(I^j)\sum_{i\geq 0}(\frac{1}{2})^{i+2} = \frac{1}{2}\,\mu(I^j)$. The remaining points in I^j are all exceptional, so $\mu(E(I^j)) \geq \mu(I^j) - \frac{1}{2}\,\mu(I^j) = \frac{1}{2}\,\mu(I^j)$. □

Lemma 8.6. *Let $x \in I^j$ be an exceptional point for some $j \leq n$. Then $\tilde{\gamma}(x) = w^j_0$, and for any $\epsilon > 0$ and any $w_k \in W_j$ there is a terminal cube T contained in the ϵ-neighborhood of x with $\tilde{\gamma}(T) = w_k$.*

Proof. Since $x \in I^j$ is exceptional, it belongs to I^j_i for each $i \in \mathbb{N}$. We can pick i large enough so that $\frac{1}{2^i} < \frac{\epsilon}{2}$. But then in the notations above, there exists a point $y \in \Omega_i$ such that $d(x,y) < \frac{\epsilon}{2}$. The statement now follows from the Basic Construction, since for each $w_k \in W_j$ there is a terminal cube T_k in the $\frac{1}{2^i}$-neighborhood of y (and so also in the $\frac{\epsilon}{2}$-neighborhood of y) with $\tilde{\gamma}(T_k) = w_k$. □

Construction of the maps, γ_l. In the basic construction we defined a map $\tilde{\gamma} : X \to W$ that we will use in order to construct a sequence of 'approximation' maps, $\gamma_1, \gamma_2, \gamma_3, \ldots\ldots$, where $\gamma_1 = \tilde{\gamma}$. In the end, we will construct the needed map, γ, as the limit (appropriately defined) of these approximation maps. We begin by putting $\gamma_1 = \tilde{\gamma}$. The *terminal cubes of* γ_1 and the *exceptional points of* γ_1 are the terminal cubes and exceptional points of the Basic Construction. Note that each of $I^1, \ldots, I^n$ contains countably many terminal cubes of γ_1 together with exceptional points that don't belong to any terminal cube.

Assume that γ_l is defined and that for each terminal cube T of γ_l, all points in the interior of T are mapped by γ_l to a single element in W, which we denote by $\gamma_l(T)$. Moreover, assume that:

(i) $\gamma_l \circ F = G \circ \gamma_l$

(ii) for any terminal cube T of γ_l in I^j, F maps T into some terminal cube T' of γ_l in I^{j+1}, for $j < n$

where F is again the embedding $(x_1, \ldots, x_j) \mapsto (x_1, \ldots, x_j, \frac{1}{2})$.

We now define γ_{l+1} on the interiors of the terminal cubes of γ_l. In particular, for any terminal cube T of γ_l in I^1, let $T^1 = T$ and let T^{j+1} be the terminal cube of I^{j+1} containing $F(T^j)$, for $j < n$. Then we have a system $T^1, \ldots, T^n$

exactly like the system $I^1, \ldots, I^n$ in the Basic Construction. We define γ_{l+1} on the interiors of $T^1, \ldots, T^n$ in the same way as we defined $\tilde{\gamma}$ on $I^1, \ldots, I^n$, letting $w^j_{\text{root}} = \gamma_l(T^j)$ and letting $w_1, \ldots, w_{r^j}$ be the R-successors of w^j_{root}. The only modification we need to make is a measure-theoretic one. In particular, in each of the terminal cubes T^j, we want to end up with a set of exceptional points that carries non-zero measure (this will be important for proving that the limit map we define, γ, has the M-property with respect to μ). To do this, assume γ_{l+1} has been defined on $T^1, \ldots, T^j$, and that for $k \leq j$, $\mu(E(T^k)) \geq \frac{1}{2}\mu(T^k)$, where $E(T^k)$ is the set of exceptional points in T^k. When we define γ_{l+1} on T^{j+1}, we make sure that at the first stage we remove terminal cubes with a total measure of no more than $\frac{1}{2}^{0+2}\,\mu(T^{j+1})$. At stage i where we are given T^{j+1}_i we remove terminal cubes with a total measure of no more than $(\frac{1}{2})^{i+2}\,\mu(T^{j+1})$. Again, this can be done because at each stage i we remove only a finite number of terminal cubes, so we can make the size of these cubes small enough to ensure we do not exceed the allocated measure. Thus, over countably many stages we remove from T^{j+1} a total measure of no more than $\mu(T^{j+1}) \sum_{i\geq 0} (\frac{1}{2})^{i+2} = \frac{1}{2}\,\mu(T^{j+1})$. Letting $E(T^{j+1})$ be the set of exceptional points in T^{j+1}, we have $\mu(E(T^{j+1})) \geq \frac{1}{2}\,\mu(T^{j+1})$.

We do this for each terminal cube T of γ_l in I^1. Next we do the same for all the remaining terminal cubes T of γ_l in I^2 (i.e. those terminal cubes in I^2 that are disjoint from D_2), and again, for all the remaining terminal cubes T of γ_l in I^3 (the terminal cubes in I^3 that are disjoint from D_3), etc. At the end of this process we have defined γ_{l+1} on the interior of each terminal cube of γ_l. For any point $x \in X$ that does not belong to the interior of any terminal cube of γ_l, we put $\gamma_{l+1}(x) = \gamma_l(x)$. The terminal cubes of γ_{l+1} are the terminal cubes of the Basic Construction applied to each of the terminal cubes of γ_l. The points in the interior of terminal cubes of γ_l that do not belong to the interior of any terminal cube of γ_{l+1} are the exceptional points of γ_{l+1}.

In view of the measure-theoretic modifications we made above, we have the following analog of Lemma 8.5:

Lemma 8.7. *Let $l \in \mathbb{N}$ and let T be any terminal cube of γ_l and $E(T)$ be the set of exceptional points of γ_{l+1} in T. Then*

$$\mu(E(T)) \geq \frac{1}{2}\,\mu(T)$$

Furthermore, the reader can convince himself that we have the following analog of Lemma 8.6 for the maps γ_l:

Lemma 8.8. *Let x be an exceptional point of γ_l and let $\gamma_l(x) = w$. Then for any $\epsilon > 0$ and any v such that wRv, there is a terminal cube T of γ_l contained in the ϵ-neighborhood of x with $\gamma_l(T) = v$.*

Finally, note that if x is an exceptional point of γ_l for some l, then $\gamma_l(x) = \gamma_{l+k}(x)$ for any $k \in \mathbb{N}$. We let B denote the set of points that are exceptional for some γ_l, and define the map $\gamma : B \to W$ as follows:

$$\gamma(x) = \lim_{l\to\infty} \gamma_l(x)$$

Lemma 8.9. $\mu(B) = \mu(X)$.

Proof. Let T_l be the set of all points that belong to some terminal cube of γ_l. Note that $T_l \supseteq T_{l+1}$ for $l \in \mathbb{N}$, and $\mu(T_1)$ is finite. Thus $\mu(\bigcap_l T_l) = \lim_{l\to\infty} \mu(T_l) = 0$. (The limit value follows from Lemma 8.7.) Finally, note that $B = X - \bigcap_l T_l$. So B is Borel, and $\mu(B) = \mu(X) - \mu(\bigcap_l T_l) = \mu(X)$. □

We have constructed a map $\gamma : B \to W$ where $\mu(B) = \mu(X)$. Moreover, by the Basic Construction, we have $\gamma_l \circ F(x) = G \circ \gamma_l(x)$ for each $l \in \mathbb{N}$. It follows that $\gamma \circ F(x) = G \circ \gamma(x)$ for $x \in B$. All that is left to show is that (i) γ is continuous, open, and surjective; and (ii) γ has the M-property with respect to μ.

Lemma 8.10. *γ has the M-property with respect to μ.*

Proof. We show that for any subset $S \subseteq W$, (i) $\gamma^{-1}(S)$ is Borel; and (ii) for any open set $O \subseteq X$, if $\gamma^{-1}(S) \cap O \neq \emptyset$ then $\mu(\gamma^{-1}(S) \cap O) \neq 0$. Note that since W is finite, it is sufficient to prove this for the case where $S = \{w\}$ for some $w \in W$.

(i) Note that $x \in \gamma^{-1}(w)$ iff x is exceptional for some γ_l and x belongs to some terminal cube T of γ_{l-1}, with $\gamma_{l-1}(T) = w$. There are only countably many such cubes, and the set of exceptional points in each such cube is closed. So $\gamma^{-1}(w)$ is a countable union of closed sets, hence Borel.

(ii) Suppose that O is open in X with $\gamma^{-1}(w) \cap O \neq \emptyset$. Let $x \in \gamma^{-1}(w) \cap O$. Again, x is exceptional for some γ_l. Pick $\epsilon > 0$ such that the ϵ-neighborhood of x is contained in O. By Lemma 8.8, there is a terminal cube T of γ_l contained in the ϵ-neighborhood of x such that $\gamma_l(T) = w$ (since wRw). Letting $E(T)$ be the set of exceptional points of γ_{l+1} in T, we know that $E(T) \subseteq \gamma^{-1}(w)$. But by Lemma 8.7, $\mu(E(T)) \geq \frac{1}{2}\mu(T) > 0$. So $E(T)$ is a subset of $\gamma^{-1}(w) \cap O$ of non-zero measure, and $\mu(\gamma^{-1}(w) \cap O) > 0$.

□

In what follows, for any $w \in W$, let $U_w = \{v \in W \mid wRv\}$ (*i.e.*, U_w is the smallest open set in W containing w).

Lemma 8.11. *γ is continuous.*

Proof. Let U be an open set in W and suppose that $x \in \gamma^{-1}(U)$. Let $\gamma(x) = w \in U$. Then x is exceptional for some γ_l. So x belongs to an (open) terminal cube T of γ_{l-1} with $\gamma_{l-1}(T) = w$. By R-monotonicity of $\langle \gamma_l(y) \rangle$ for all $y \in B$, we know that for any $y \in T$, $\gamma(y) \in U_w$—*i.e.*, $T \subseteq \gamma^{-1}(U_w)$. Moreover, since $w \in U$ and U is open, we have $U_w \subseteq U$. Thus $x \in T \subseteq \gamma^{-1}(U)$. This shows that $\gamma^{-1}(U)$ is open in X. □

Lemma 8.12. *γ is open.*

Proof. Let O be open in B and let $w \in \gamma(O)$. We show that $U_w \subseteq \gamma(O)$. We know that there exists $x \in O$ such that $\gamma(x) = w$. Moreover, x is exceptional for some γ_l. Pick $\epsilon > 0$ small enough so that the ϵ-neighborhood of x is contained in O. By Lemma 8.8, for each $v \in U_w$ there is a terminal cube T_v of γ_l contained in the ϵ-neighborhood of x such that $\gamma_l(T_v) = v$. But then for any exceptional point y_v of γ_{l+1} that lies in T_v, we have $\gamma(y_v) = \gamma_{l+1}(y_v) = v$, and $y_v \in O$. We have shown that for all $v \in U_w$, $v \in \gamma(O)$. It follows that $\gamma(O)$ is open. □

Lemma 8.13. *γ is surjective.*

Proof. This follows immediately from the fact that γ 'hits' each of the roots, $w_0^1, \dots, w_0^{n+1}$, of K and γ is open. □

Corollary 8.14. *ϕ is refuted in $\mathcal{M}$.*

Proof. We stipulated that ϕ is refuted in the dynamic Kripke model $K = \langle W, R, G, V_1 \rangle$. Equivalently, letting $M_1 = \langle A_K, h_G, V_1 \rangle$ be the dynamic algebraic model corresponding to K, ϕ is refuted in M_1. By Lemma 8.11, Lemma 8.12, Lemma 8.13, and Lemma 8.10, we showed that $\gamma : X \to W$ is (i) continuous, open and surjective; (ii) $\gamma \circ f = G \circ \gamma$; and (iii) γ has the M-property with respect to μ. Thus by Lemma 7.5, the map $\Phi : \langle A_K, h_G \rangle \to \langle \mathcal{M}_X^\mu, h_F^{|\cdot|} \rangle$ defined by

$$\Phi(S) = |\gamma^{-1}(S)|$$

is a dynamic embedding. We now define the valuation $V_2 : PV \to \mathcal{M}_X^\mu$ by:

$$V_2(p) = \Phi \circ V_1(p)$$

and we let $M_2 = \langle \mathcal{M}_X^\mu, h_F^{|\cdot|}, V_2 \rangle$. By Corollary 7.3, $M_2 \not\models \phi$. In view of the isomorphism $M_X^\mu \cong \mathcal{M}$, we have shown that ϕ is refuted in $\mathcal{M}$. □

We have shown that for *any* formula $\phi \notin S4C$, ϕ is refuted in $\mathcal{M}$. We conclude the section by stating this completeness result more formally as follows:

Theorem 8.15. *$DML_{\mathcal{M}} \subseteq S4C$.*

9 Completeness for a single measure model

In this section we prove a strengthening of the completeness result of the previous section, showing that there is a single dynamic measure model $\langle \mathcal{M}, h, V \rangle$ in which every non-theorem of $S4C$ is refuted.

Definition 9.1. *Denote by $\mathcal{M}^\omega$ the product $\mathcal{M} \times \mathcal{M} \times \mathcal{M} \dots$ This is a Boolean algebra, where Boolean operations are defined component-wise:*

$$\begin{aligned}(a_1, a_2, a_3, \dots) \vee (b_1, b_2, b_3, \dots) &= (a_1 \vee b_1, a_2 \vee b_2, a_3 \vee b_3, \dots)\\ (a_1, a_2, a_3, \dots) \wedge (b_1, b_2, b_3, \dots) &= (a_1 \wedge b_1, a_2 \wedge b_2, a_3 \wedge b_3, \dots)\\ -(a_1, a_2, a_3, \dots) &= (-a_1, -a_2, -a_3, \dots)\end{aligned}$$

Definition 9.2. *We say $(a_1, a_2, a_3, \dots)$ is an* **open** *element in $\mathcal{M}^\omega$ if a_k is open in $\mathcal{M}$ for each $k \in \mathbb{N}$.*

The collection of open elements in $\mathcal{M}^\omega$ is closed under finite meets, arbitrary joins and contains the top and bottom element (since operations in $\mathcal{M}^\omega$ are componentwise). We define the operator I_ω on $\mathcal{M}^\omega$ by:

$$I_\omega(a_1, a_2, a_3, \dots) = (Ia_1, Ia_2, Ia_3, \dots)$$

Then I_ω is an interior operator on $\mathcal{M}^\omega$ (the proof is the same as the proof of Lemma 5.8). So the algebra $\mathcal{M}^\omega$ together with the interior operator I_ω is a topological Boolean algebra.

Lemma 9.3. *There is a dynamic algebraic model $M = \langle \mathcal{M}^\omega, h, V \rangle$ such that for any formula $\phi \in L_{\Box,\bigcirc}$, the following are equivalent:*

(i) $S4C \vdash \phi$;

(ii) $M \models \phi$.

Proof. Let $\langle \phi_k \rangle$ be an enumeration of all non-theorems of $S4C$ (there are only countably many formulas, so only countably many non-theorems). By completeness of $S4C$ for $\mathcal{M}$, for each $k \in \mathbb{N}$, there is a model $M_k = \langle \mathcal{M}, h_k, V_k \rangle$ such that $M_k \not\models \phi_k$. We construct a model $M = \langle \mathcal{M}^\omega, h, V \rangle$, where h and V are defined as follows. For any $\langle a_k \rangle_{k \in \mathbb{N}} = (a_1, a_2, a_3, \dots) \in \mathcal{M}^\omega$, and for any propositional variable p:

$$h((a_1, a_2, a_2, \dots)) = \langle h_k(a_k) \rangle_{k \in \mathbb{N}}$$

$$V(p) = \langle V_k(p) \rangle_{k \in \mathbb{N}}$$

(The fact that h is an O-operator follows from the fact that h is computed componentwise according to the h_k's, and each h_k is an O-operator).

We can now prove the lemma. The direction $(i) \Rightarrow (ii)$ follows from Proposition 4.7. We show $(ii) \Rightarrow (i)$, by proving the contrapositive. Suppose that $S4C \not\models \phi$. Then $\phi = \phi_k$ for some $k \in \mathbb{N}$. We claim that

$$\pi_k V(\phi) = V_k(\phi)$$

where π_k is the projection onto the kth coordinate. (Proof: By induction on complexity of ϕ, and the fact that π_k is a topological homomorphism.) In particular, $\pi_k V(\phi_k) = V_k(\phi_k) \neq 1$. So $V(\phi_k) \neq 1$, and $M \not\models \phi_k$. □

Lemma 9.4. *$\mathcal{M}^\omega$ is isomorphic to $\mathcal{M}$.*

Proof. We need to construct an isomorphism from $\mathcal{M}^\omega$ onto $\mathcal{M}$. Let $(a_1, a_2, a_3, \dots)$ be an arbirary element in $\mathcal{M}^\omega$. Then for each $k \in \mathbb{N}$, we can choose a set $A_k \subseteq [0, 1]$ such that $a_k = |A_k|$ and $1 \notin A_k$. We define a sequence of points s_k in the real interval [0, 1] as follows:

$$s_0 = 0$$
$$s_1 = 1/2$$
$$s_2 = 3/4$$

In general, $s_k = \frac{2^k - 1}{2^k}$ $(k \geq 1)$. We now define a sequence of intervals I_k having the a_k's as endpoints:

$$I_0 = [0, \frac{1}{2})$$
$$I_1 = [\frac{1}{2}, \frac{3}{4})$$
$$I_2 = [\frac{3}{4}, \frac{7}{8})$$

and in general $I_k = [s_k, s_{k+1})$. Our idea is to map each set A_k into the interval I_k. We do this by letting $B_k = l_k\, A_k + s_k$ where l_k is the length of I_k. Clearly $B_k \subseteq I_k$ and $B_k \cap B_j = \emptyset$ for all $k \neq j$. We can now define the map $h : \mathcal{M}^\omega \to \mathcal{M}$ by:

$$h(a_1, a_2, a_3, \dots) = |\bigcup_{k \in \mathbb{N}} B_k\,|$$

where B_k is defined as above. The reader can now verify that h is an isomorphism. □

Corollary 9.5. *There is a dynamic measure model* $M = \langle \mathcal{M}, h, V \rangle$ *such that for any formula* $\phi \in L_{\Box,\bigcirc}$*, the following are equivalent:*

(*i*) *S4C* $\vdash \phi$*;*

(*ii*) $M \models \phi$*.*

Proof. Immediate from Lemma 9.3 and Lemma 9.4. □

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