Modal Probability Logic

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Wes Holliday and Thomas Icard
Berkeley and Stanford

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References

- M. Fattorosi-Barnaba and G. Amati, "Modal Operators with Probabilistic Interpretations," 1987.
- Wiebe van der Hoek, "Some Considerations on the Logic PDF," 1992.
- Wiebe van der Hoek and John-Jules Meyer, "Modalities for Reasoning about Knowledge and Uncertainties," 1996.
- Chunlai Zhou, "A Complete Deductive System for Probability Logic," 2007.

Language

Definition

Given a countable set $\mathrm{At} = \{p, q, r, \dots\}$ of atomic sentences, we define the language $\mathcal{L}(\mathrm{P})$ as follows

$$\varphi ::= p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid \mathrm{P}_r^{>}\varphi,$$

where $p \in \mathrm{At}$ and $r \in \mathbb{R}$. We read $\mathrm{P}_r^{>}\varphi$ as "the probability of φ is strictly greater than r." There is an obvious multi-agent extension.

We define $\vee$, $\rightarrow$, and $\leftrightarrow$ as usual, and:

- $\mathrm{P}_r^{\geq}\varphi := \neg\mathrm{P}_{1-r}^{>}\neg\varphi$ and $\mathrm{P}_r^{<}\varphi := \mathrm{P}_{1-r}^{>}\neg\varphi$;
- $\mathrm{P}_r^{\leq}\varphi := \neg\mathrm{P}_{r-1}^{<}\neg\varphi$;
- $\mathrm{P}_r^{=}\varphi := \neg\mathrm{P}_r^{>}\varphi \wedge \neg\mathrm{P}_r^{<}\varphi$;
- $\Box\varphi := \mathrm{P}_1^{\geq}\varphi$.

Bases

Definition (Base)

A *base* is a set $F \subseteq [0,1]$ such that:

1. $\{0,1\} \subseteq F$;
2. if $r, s \in F$ and $r + s \leq 1$, then $r + s \in F$;
3. if $r \in F$, then $1 - r \in F$.

Models

Definition (Probabilistic Kripke Model over F)

A *probabilistic Kripke model over base F* is a tuple $\mathcal{M} = \langle W, R, \{\mu_w\}_{w \in W}, V\rangle$ such that:

1. W is a nonempty set;
2. R is a *serial* binary relation on W, i.e., such that for all $w \in W$, $R(w) = \{v \in W \mid wRv\} \neq \emptyset$;
3. $\mu_w \colon \wp(R(w)) \to F$ such that $\mu_w(R(w)) = 1$ and if $A \cap B = \emptyset$, then $\mu_w(A \cup B) = \mu_w(A) + \mu_w(B)$.

If $F = [0, 1]$, then $\mathcal{M}$ is simply a *probabilistic Kripke model*.

Semantics

Definition (Truth and Consequence)

Where $\mathcal{M}$ is a probabilistic Kripke model over base F and $\varphi \in \mathcal{L}(\mathrm{P})$, we define $\mathcal{M}, w \vDash \varphi$ ("φ is true at w in $\mathcal{M}$") by:

- $\mathcal{M}, w \vDash p$ iff $w \in V(p)$;
- $\mathcal{M}, w \vDash \neg\varphi$ iff $\mathcal{M}, w \nvDash \varphi$;
- $\mathcal{M}, w \vDash (\varphi \wedge \psi)$ iff $\mathcal{M}, w \vDash \varphi$ and $\mathcal{M}, w \vDash \psi$;
- $\mathcal{M}, w \vDash \mathrm{P}_r^{>}\varphi$ iff $\mu_w(\llbracket\varphi\rrbracket_w^{\mathcal{M}}) > r$,

where $\llbracket\varphi\rrbracket_w^{\mathcal{M}} = \{v \in R(w) \mid \mathcal{M}, v \vDash \varphi\}$.

The definition of consequence is standard: given a class $\mathcal{C}$ of probabilistic Kripke models over bases, $\Gamma \subseteq \mathcal{L}(\mathrm{P})$, and $\varphi \in \mathcal{L}(\mathrm{P})$, we have $\Gamma \vDash_{\mathcal{C}} \varphi$ ("φ is a consequence of Γ over $\mathcal{C}$") iff for all $\mathcal{M}$ in $\mathcal{C}$ and w in $\mathcal{M}$, if $\mathcal{M}, w \vDash \gamma$ for all $\gamma \in \Gamma$, then $\mathcal{M}, w \vDash \varphi$.

Compactness

For a finite base F and the class $\mathcal{C}$ of all probabilistic Kripke models over F, the consequence relation $\vDash_{\mathcal{C}}$ is compact.

However, for the class $\mathcal{C}$ of all probabilistic Kripke models (i.e., $F = [0, 1]$), the consequence relation $\vDash_{\mathcal{C}}$ is not compact.

Simply consider the set

$$\Gamma = \{\neg P_r^{=} q \mid r \in \mathbb{R}\}.$$

Γ is clearly finitely satisfiable but not satisfiable in the class of probabilistic Kripke models.

By contrast, Γ is not finitely satisfiable in the class of probabilistic Kripke models over a finite base F. Take $\Gamma_0 = \{\neg P_r^{=} q \mid r \in F\}$.

Logic

For a finite base $F = \{r_0, \dots, r_n\}$ with $r_0 < r_1 \dots r_{n-1} < r_n$, **PFD**$_F$ has the following axioms and rules for $r, s \in \mathbb{R}$:

- Taut: all propositional tautologies as axioms;
- MP: if $\vdash \varphi$ and $\vdash \varphi \to \psi$, then $\vdash \psi$;
- Nec: if $\vdash \varphi$, then $\vdash \Box\varphi$;
- $\mathrm{P}^{\geq}_{0}\varphi$; $\quad \mathrm{P}^{>}_{r}\varphi \to \mathrm{P}^{\geq}_{r}\varphi$; $\quad \mathrm{P}^{\geq}_{r}\varphi \to \mathrm{P}^{>}_{s}\psi$ for $r > s$;
- $\Box(\varphi \to \psi) \to [(\mathrm{P}^{>}_{r}\varphi \to \mathrm{P}^{>}_{r}\psi) \wedge (\mathrm{P}^{\geq}_{r}\varphi \to \mathrm{P}^{\geq}_{r}\psi)]$;
- $\mathrm{P}^{>}_{r+s}(\varphi \vee \psi) \to (\mathrm{P}^{>}_{r}\varphi \vee \mathrm{P}^{\geq}_{s}\psi)$ for $r + s \in [0, 1]$;
- $\Box\neg(\varphi \wedge \psi) \to \big((\mathrm{P}^{>}_{r}\varphi \wedge \mathrm{P}^{\geq}_{s}\psi) \to \mathrm{P}^{>}_{r+s}(\varphi \vee \psi)\big)$ for $r + s \in [0, 1]$;
- $\mathrm{P}^{>}_{r_m}\varphi \to \mathrm{P}^{\geq}_{r_{m+1}}\varphi$ for $0 \leq m < n$.

Completeness & Decidability

Let's say that a logic **L** is *strongly complete* with respect to a class $\mathcal{C}$ of models iff for all $\Gamma \subseteq \mathcal{L}(\mathrm{P})$ and $\varphi \in \mathcal{L}(\mathrm{P})$, if $\Gamma \vDash_{\mathcal{C}} \varphi$, then for some finite $\Gamma_0 \subseteq \Gamma$, $\vdash_{\mathrm{P}} (\bigwedge \Gamma_0) \rightarrow \varphi$.

Theorem (Fattorosi-Barnaba & Amati 1989, van der Hoek 1992)

For any finite base F, the logic $\mathbf{PFD}_F$ is sound and strongly complete with respect to the class of all probabilistic Kripke models over F.

Theorem (van der Hoek 1992)

For any finite base F, the logic $\mathbf{PFD}_F$ has the finite model property with respect to that class and is decidable.

Comparison with Zhou (2009)

Zhou (2009) works with a language that extends that of propositional logic with modal operators $P_r^{\geq}$ for $r \in \mathbb{Q} \cap [0, 1]$.

Zhou's models are a generalization of probabilistic Kripke models with σ-algebras for the domains of the measures. The co-domains of the measures are always $[0, 1]$, not a restricted base F. The truth clauses are the same as we gave before.

Zhou's consequence relation is not compact. He obtains a sound and complete proof system using an infinitary rule:

- Arch: if $\vdash \varphi \to P_s^{\geq}\psi$ for all $s < r$, then $\vdash \varphi \to P_r^{\geq}\psi$.

Zhou then confirms a conjecture of Larry Moss that Arch (for "Archimedean") can be replaced by a finitary rule.