Recent Work in Epistemic Logic

Logic Group Colloquium

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March 1, 2013
Outline

▶ Preliminaries
▶ Basic Epistemic Logic
  • Modeling Knowledge
  • Common Knowledge
  • Information Update
▶ Mathematical Themes
  • Preservation Theorems
  • Uniform Substitution
  • Relations to FOL
▶ Conclusion
What is Epistemic Logic?

Epistemic logic provides a formal framework for modeling the knowledge of agents. Used by philosophers, theoretical computer scientists, AI researchers, game theorists, and others, epistemic logic has become one of the main application areas for modal logic.
A Very Brief History

12-15th c. Informal EL investigated by logicians of the Middle Ages.
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90s-now Dynamic epistemic logic is developed in CS (Jan Plaza) and logic, especially by the “Amsterdam School” around Johan van Benthem.
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90s-now Dynamic epistemic logic is developed in CS (Jan Plaza) and logic, especially by the “Amsterdam School” around Johan van Benthem.
Now I am interested in bringing epistemic logic back into contact with philosophy, using it as a tool to formalize existing theories of knowledge and develop new and improved ones. See, for example:

Wesley Holliday. Forthcoming.
“Epistemic Logic and Epistemology.” *Handbook of Formal Philosophy*.

Wesley Holliday. Forthcoming.
“Epistemic Closure and Epistemic Logic I.” *Journal of Philosophical Logic*.

Wesley Holliday. 2012.
“Epistemic Logic, Relevant Alternatives, and the Dynamics of Context.”
*New Directions in Logic, Language, and Computation, LNCS*.

Wesley Holliday and John Perry. Forthcoming.
“Roles, Rigidity, and Quantification in Epistemic Logic.” *Trends in Logic*.
Logic and Philosophy

The project of proving results about formalized philosophical theories has led to the development of new logical techniques.
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This paper demonstrates an alternative method of proving completeness theorems in modal logic, without the standard Henkin-style canonical model construction, that yields results on satisfiability in finite models and complexity almost for free.
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This paper demonstrates an alternative method of proving completeness theorems in modal logic, without the standard Henkin-style canonical model construction, that yields results on satisfiability in finite models and complexity almost for free.

I am working to show how this method gives simple completeness proofs for standard systems of modal logic—good for students.
Today’s Topic

But my talk today won’t presuppose familiarity with any philosophical theories or standard techniques in modal logic.
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Today I’ll deal with propositional epistemic logic, but first-order epistemic logic is also philosophically and technically rich.
Definition (Propositional Epistemic Language)

Given sets \( \text{At} = \{p, q, r, \ldots \} \) and \( \text{Agt} = \{a, b, c, \ldots \} \) with \( |\text{Agt}| = \kappa \), the epistemic language \( \mathcal{L}^\kappa_{\text{EL}} \) is generated by

\[
\phi ::= p \mid \neg \phi \mid (\phi \land \phi) \mid K_a \phi.
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We read $K_a \varphi$ as "agent $a$ knows that $\varphi$.

Let $\hat{K}_a \varphi := \neg K_a \neg \varphi$ for "$\varphi$ is consistent with $a$’s knowledge."
**Definition (Propositional Epistemic Language)**

Given sets $A_t = \{p, q, r, \ldots\}$ and $A_{gt} = \{a, b, c, \ldots\}$ with $|A_{gt}| = \kappa$, the epistemic language $\mathcal{L}_E^\kappa$ is generated by

$$\varphi ::= p \mid \neg \varphi \mid (\varphi \land \varphi) \mid K_a \varphi.$$  

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Given a finite set $A \subseteq A_{gt}$, let

$$E_A \varphi := \bigwedge_{a \in A} K_a \varphi,$$

for “everyone in $A$ knows that $\varphi$.”
Definition (Propositional Epistemic Language)

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E_A \varphi := \bigwedge_{a \in A} K_a \varphi,
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for "everyone in \( A \) knows that \( \varphi \)."

Where \( \square \) is \( K_a \) or \( E_A \), \( \square^1 \varphi := \square \varphi \) and \( \square^{n+1} \varphi := \square \square^n \varphi \) (\( n \geq 1 \)).
Definition (Epistemic Frames and Models)

A $\kappa$-agent epistemic model is a triple $\mathcal{M} = \langle W, \{R_a\}_{a \in \text{Agt}}, V \rangle$:
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- $W$ is a non-empty set (of “possibilities,” “scenarios,” “states,” or “worlds”);
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- for each $a \in \text{Agt}$ ($|\text{Agt}| = \kappa$), $R_a$ is a binary relation on $W$; $R_a$ can be thought of as representing agent $a$’s **uncertainty**.
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- for each $a \in \text{Agt}$ ($|\text{Agt}| = \kappa$), $R_a$ is a binary relation on $W$; let’s call $R_a$ agent $a$’s “epistemic accessibility” relation; $wR_a v$ means that everything $a$ knows in $w$ is true in $v$. 

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- $wR_a v$ means that $v$ is compatible with $a$’s knowledge in $w$. 

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Definition (Epistemic Model)

A \( \kappa \)-agent epistemic **model** is a triple \( \mathcal{M} = \langle W, \{ R_a \}_{a \in \text{Agt}}, V \rangle \):

- \( V : \text{At} \rightarrow \wp(w) \) assigns to each atomic sentence the set of scenarios in which it holds. We say \( \mathcal{M} \) is **based on** \( \mathcal{F} \).
Three Ideas for the Semantics

The semantics will be based on three ideas from philosophy:

1. The knowledge operator $K_a$ can be interpreted as a restricted quantifier ranging over those possibilities compatible with $a$'s knowledge (J. Hintikka, *Knowledge and Belief*, 1962).

2. It is common knowledge that $\phi$ when everyone knows that $\phi$, everyone knows that everyone knows that $\phi$, everyone knows that everyone knows that everyone knows that $\phi$, etc. (inspired by D. Lewis, *Convention*, 1969).

3. Knowledge acquisition (information update) can be understood in terms of the elimination of possibilities from an agent's initial epistemic state (R. Stalnaker, *Inquiry*, 1987).
Basic Epistemic Logic

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Idea 1: Knowledge Operators as Restricted Quantifiers

Example (Berkeley and Copenhagen)

Let $K_b$ stand for agent $b$ knows that and $K_c$ stand for agent $c$ knows that. Suppose agent $b$, who lives in Berkeley, knows that agent $c$ lives in Copenhagen. Let $r$ stand for ‘it’s raining in Copenhagen’. Although $b$ doesn’t know whether it’s raining in Copenhagen, $b$ knows that $c$ knows whether it’s raining there:
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$$\neg (K_b r \lor K_b \neg r) \land K_b (K_c r \lor K_c \neg r).$$
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$$\neg(K_b r \lor K_b \neg r) \land K_b (K_c r \lor K_c \neg r).$$

The following picture depicts a situation in which this is true, where an arrow represents compatibility with one’s knowledge:
Now suppose that agent \(b\) doesn’t know whether agent \(c\) has left Copenhagen for a vacation. (Let \(v\) stand for ‘\(c\) has left Copenhagen on vacation’.) Agent \(b\) knows that if \(c\) is not on vacation, then \(c\) knows whether it’s raining in Copenhagen; but if \(c\) is on vacation, then \(c\) won’t bother to follow the weather.

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K_b(\neg v \rightarrow (K_c r \lor K_c \neg r)) \land K_b(v \rightarrow \neg (K_c r \lor K_c \neg r)).
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Idea 1: Knowledge Operators as Restricted Quantifiers

Definition (Truth)
Given a $\kappa$-agent model $\mathcal{M} = \langle W, \{ R_a \}_{a \in \text{Agt}}, V \rangle$ with $w \in W$ and $\varphi \in \mathcal{L}_{\kappa}^{\text{EL}}$, we define $\mathcal{M}, w \models \varphi$ (“$\varphi$ is true in $\mathcal{M}$ at $w$”) as follows:

- $\mathcal{M}, w \models p$ iff $w \in V(p)$;
- $\mathcal{M}, w \models \neg \varphi$ iff $\mathcal{M}, w \not\models \varphi$;
- $\mathcal{M}, w \models (\varphi \land \psi)$ iff $\mathcal{M}, w \models \varphi$ and $\mathcal{M}, w \models \psi$. 

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\begin{align*}
\mathcal{M}, w \models p & \quad \text{iff} \quad w \in V(p); \\
\mathcal{M}, w \models \neg \varphi & \quad \text{iff} \quad \mathcal{M}, w \not\models \varphi; \\
\mathcal{M}, w \models (\varphi \land \psi) & \quad \text{iff} \quad \mathcal{M}, w \models \varphi \text{ and } \mathcal{M}, w \models \psi; \\
\mathcal{M}, w \models K_a \varphi & \quad \text{iff} \quad \forall v \in W: \text{ if } wR_a v \text{ then } \mathcal{M}, v \models \varphi;
\end{align*}
\]
Now suppose that agent $b$ doesn’t know whether agent $c$ has left Copenhagen for a vacation. (Let $v$ stand for ‘$c$ has left Copenhagen on vacation’.) Agent $b$ knows that if $c$ is not on vacation, then $c$ knows whether it’s raining in Copenhagen; but if $c$ is on vacation, then $c$ won’t bother to follow the weather.

$$K_b(\neg v \rightarrow (K_c r \lor K_c \neg r)) \land K_b(v \rightarrow \neg (K_c r \lor K_c \neg r)).$$
Idea 1: Knowledge Operators as Restricted Quantifiers

Definition (Truth and Validity)

Given a $\kappa$-agent model $\mathcal{M} = \langle W, \{ R_a \}_{a \in \text{Agt}}, V \rangle$ with $w \in W$ and $\varphi \in \mathcal{L}_{\text{EL}}^\kappa$, we define $\mathcal{M}, w \models \varphi$ as follows (other cases as before):

$$\mathcal{M}, w \models K_a \varphi \quad \text{iff} \quad \forall v \in W: \text{if } w R_a v \text{ then } \mathcal{M}, v \models \varphi;$$

We say $\varphi$ is globally true in $\mathcal{M}$ iff for all $w \in W$, $\mathcal{M}, w \models \varphi$;

$\varphi$ is valid on frame $F$ iff globally true in all models based on $F$;

$\varphi$ is valid on a class $C$ of models (resp. a class $F$ of frames) iff globally true in all models in $C$ (resp. based on frames in $F$).
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We say $\varphi$ is **globally true** in $\mathcal{M}$ iff for all $w \in W$, $\mathcal{M}, w \models \varphi$;
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$\mathcal{M}, w \vDash K_a \varphi$ iff $\forall v \in W: \text{if } wR_a v \text{ then } \mathcal{M}, v \vDash \varphi$;

We say $\varphi$ is globally true in $\mathcal{M}$ iff for all $w \in W$, $\mathcal{M}, w \vDash \varphi$;

$\varphi$ is valid on frame $\mathcal{F}$ iff globally true in all models based on $\mathcal{F}$;

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Definition (Truth and Validity)

Given a \( \kappa \)-agent model \( \mathcal{M} = \langle W, \{ R_a \}_{a \in \text{Agt}}, V \rangle \) with \( w \in W \) and \( \varphi \in \mathcal{L}_\text{EL}^\kappa \), we define \( \mathcal{M}, w \models \varphi \) as follows (other cases as before):

\[
\mathcal{M}, w \models K_a \varphi \quad \text{iff} \quad \forall \nu \in W: \text{if } wR_a \nu \text{ then } \mathcal{M}, \nu \models \varphi;
\]

We say \( \varphi \) is **globally true** in \( \mathcal{M} \) iff for all \( w \in W \), \( \mathcal{M}, w \models \varphi \);

\( \varphi \) is **valid on frame** \( \mathcal{F} \) iff globally true in all models based on \( \mathcal{F} \);

\( \varphi \) is **valid on a class** \( \mathcal{C} \) of models (resp. a class \( \mathcal{I} \) of frames) iff globally true in all models in \( \mathcal{C} \) (resp. based on frames in \( \mathcal{I} \)).

For language \( \mathcal{L} \), the **\( \mathcal{L} \)-theory** of a class \( \mathcal{C} \) of models, \( \text{Th}_\mathcal{L}(\mathcal{C}) \) (resp. class \( \mathcal{I} \) of frames) is the set of \( \varphi \in \mathcal{L} \) valid on \( \mathcal{C} \) (resp. \( \mathcal{I} \)).
Correspondence

Proposition

For any frame $\mathcal{F} = \langle W, \{ R_a \}_{a \in \text{Agt}} \rangle$:

- $\top K_a \varphi \rightarrow \varphi$ is valid on $\mathcal{F}$ iff $R_a$ is reflexive;
- $\neg K_a \varphi$ is valid on $\mathcal{F}$ iff $R_a$ is symmetric;
- $\neg K_a \varphi \rightarrow K_a \neg K_a \varphi$ is valid on $\mathcal{F}$ iff $R_a$ is Euclidean.

Cf. Theorem 3.5 of Bjarni Jonsson and Alfred Tarski. 1951. "Boolean Algebras with Operators."
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1. $\text{T } K_a \varphi \rightarrow \varphi$ is valid on $\mathcal{F}$ iff $R_a$ is reflexive;

4. $K_a \rightarrow K_aK_a\varphi$ is valid on $\mathcal{F}$ iff $R_a$ is transitive;

B. $\neg \varphi \rightarrow K_a\neg K_a\varphi$ is valid on $\mathcal{F}$ iff $R_a$ is symmetric;
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2. $\mathbf{K}_a \rightarrow K_a K_a \varphi$ is valid on $\mathcal{F}$ iff $R_a$ is transitive;

3. $\mathbf{B} \quad \neg \varphi \rightarrow K_a \neg K_a \varphi$ is valid on $\mathcal{F}$ iff $R_a$ is symmetric;

4. $\neg K_a \varphi \rightarrow K_a \neg K_a \varphi$ is valid on $\mathcal{F}$ iff $R_a$ is Euclidean.
Basic Epistemic Logic

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3. $\text{B } \neg \varphi \rightarrow K_a \neg K_a \varphi$ is valid on $\mathcal{F}$ iff $R_a$ is symmetric;

4. $\text{5 } \neg K_a \varphi \rightarrow K_a \neg K_a \varphi$ is valid on $\mathcal{F}$ iff $R_a$ is Euclidean.

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4. $\text{5} \ \neg K_a \varphi \rightarrow K_a \neg K_a \varphi$ is valid on $\mathcal{F}$ iff $R_a$ is Euclidean.

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In CS and game theory, it is typically assumed that every $R_a$ is an equivalence relation. We call such models partition models.
Complexity

Theorem (Complexity)

1. model-checking is in linear time (in the size of $M$ and $\varphi$);
2. satisfiability in 1-agent models is $\text{PSPACE}$-complete;
3. satisfiability in 1-agent partition models is $\text{NP}$-complete;
4. satisfiability in 2-agent partition models is $\text{PSPACE}$-complete.

If we bound the nesting depth of $K$, satisfiability drops down to $\text{NP}$-complete in most cases; if we further restrict to finitely many atomic sentences, satisfiability drops down to linear time.
Completeness

**Theorem**
The $\mathcal{L}_\text{EL}^\kappa$-theory of the class of all $\kappa$-agent epistemic models is axiomatized by the system $K_\kappa$:

- all substitutions of propositional *tautologies* as axioms;
- the $K$ axiom, $K_a(\varphi \rightarrow \psi) \rightarrow (K_a \varphi \rightarrow K_a \psi)$, for $a \in \text{Agt}$;
- *modus ponens* and *necessitation*, $\vdash \varphi \Rightarrow \vdash K_a \varphi$, for $a \in \text{Agt}$.
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- modus ponens and necessitation, $\vdash \varphi \Rightarrow \vdash K_a \varphi$, for $a \in \text{Agt}$.

To axiomatize the theories of other model classes, add the axioms corresponding to the defining properties of the class:
Completeness

Theorem
The $\mathcal{L}_\text{EL}^\kappa$-theory of the class of all $\kappa$-agent epistemic models is axiomatized by the system $K_\kappa$:

- all substitutions of propositional tautologies as axioms;
- the K axiom, $K_a(\phi \to \psi) \to (K_a\phi \to K_a\psi)$, for $a \in \text{Agt}$;
- modus ponens and necessitation, $\vdash \phi \Rightarrow \vdash K_a\phi$, for $a \in \text{Agt}$.

To axiomatize the theories of other model classes, add the axioms corresponding to the defining properties of the class:

- for reflexive models, add the T axiom, $K_a\phi \to \phi$;
- for transitive models, add the 4 axiom, $K_a\phi \to K_aK_a\phi$;
- for Euclidean models, add the 5 axiom, $\neg K\phi \to K\neg K\phi_a$;
- etc.

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Idea 2: Common Knowledge as Transitive Closure

The “everyone knows” operator is not the only group knowledge operator of interest. To see this, consider the following example.

Example (Consecutive Numbers (J.E. Littlewood 1953))

Two agents, \( b \) and \( g \), are facing each other. They each see a number on the other's forehead. A perfectly trustworthy source tells them: one of you has \( n \in \mathbb{N} \) and the other has \( n + 1 \).

Let \( b_m \) stand for “the number on \( b \)'s head is \( m \),” and let \( g_m \) stand for “the number on \( g \)'s head is \( m \).” The truth is:

\[ b_2 \land g_3. \]
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Let $b_m$ stand for “the number on $b$’s head is $m$,” and let $g_m$ stand for “the number on $g$’s head is $m$.” The truth is: $b_2 \land g_3$. 
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Let $b_m$ stand for “the number on $b$’s head is $m$,” and let $g_m$ stand for “the number on $g$’s head is $m$.” The truth is: $b_2 \land g_3$.

What should an epistemic model representing this look like?
Here is an epistemic model representing the example:
Here is an epistemic model representing the example:

\[ M, w \models K b_3 \]
\[ M, w \models K (b_2 \lor b_4) \]
\[ M, w \models K K g_1 \]

\[ b_4, g_5 \]
\[ b_4, g_3 \]
\[ b_2, g_3 \]
\[ b_2, g_1 \]
\[ b_0, g_1 \]

\[ \ldots \]
Here is an epistemic model representing the example:

\[ \mathcal{M}, w \models K_b g_3; \]
Here is an epistemic model representing the example:

\[ M, w \models K_b g_3; \quad M, w \models K_b (b_2 \lor b_4); \]
Basic Epistemic Logic

Here is an epistemic model representing the example:

\[ M, w \models K_b g_3; \ M, w \models K_b (b_2 \lor b_4); \ M, w \models K_b K_g (g_1 \lor g_3 \lor g_5); \]
Here is an epistemic model representing the example:

\[ M, w \models K_b g_3; \, M, w \models K_b (b_2 \lor b_4); \, M, w \models K_b K_g (g_1 \lor g_3 \lor g_5); \, M, w \models K_b K_g K_b (g_1 \lor g_3 \lor g_5); \text{ and so on...} \]
Here is an epistemic model representing the example:

\[ \mathcal{M}, w \models \hat{K}_g g_1 \land \hat{K}_g g_3; \]
Here is an epistemic model representing the example:

\[ M, w \models \hat{K}_g b_1 \land \hat{K}_g b_3; \quad M, w \models \hat{K}_g \hat{K}_b K_g g_1; \]
Basic Epistemic Logic

Here is an epistemic model representing the example:

\[ M, w \models \mathcal{K}_g g_1 \land \mathcal{K}_g g_3; \ M, w \models \mathcal{K}_g \mathcal{K}_b \mathcal{K}_g g_1; \text{ and interestingly, } \]
\[ M, w \models E_{\{b,g\}} \neg g_5 \land \neg E_{\{b,g\}} E_{\{b,g\}} \neg g_5. \]
Here is an epistemic model representing the example:

\[ M, w \models \hat{K} g_1 \land \hat{K} g_3; M, w \models \hat{K} g \hat{K} b \hat{K} g_1; \text{ and interestingly, } M, w \models E_{\{b,g\}} \neg g_5 \land \neg E_{\{b,g\}} E_{\{b,g\}} \neg g_5. \text{ Not common knowledge.} \]
Definition (Basic Epistemic Languages)

Given countable sets $\text{At} = \{p, q, r, \ldots \}$ and $\text{Agt} = \{a, b, c, \ldots \}$ with $|\text{Agt}| = \kappa$, the epistemic language $\mathcal{L}^\kappa_{\text{EL}}$ is generated by

$$\varphi ::= p \mid \neg \varphi \mid (\varphi \land \varphi) \mid K_a \varphi.$$  

For the epistemic language with common knowledge, $\mathcal{L}^\kappa_{\text{EL-C}}$,

$$\varphi ::= p \mid \neg \varphi \mid (\varphi \land \varphi) \mid K_a \varphi \mid C_A \varphi,$$

where $A \subseteq \text{Agt}$. 

We read $C_A \varphi$ as "it is common knowledge among the members of $A$ that $\varphi$."

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where $A \subseteq \text{Agt}$. We read $C_A \varphi$ as "it is common knowledge among the members of $A$ that $\varphi$."
Idea 2: Common Knowledge as Transitive Closure

Definition
Given a $\kappa$-agent model $\mathcal{M} = \langle W, \{ R_a \}_{a \in \text{Agt}}, V \rangle$ with $w \in W$ and $\varphi \in \mathcal{L}_{\text{EL-C}}^\kappa$, we define $\mathcal{M}, w \models \varphi$ as follows (other cases as before):

- $\mathcal{M}, w \models K_a \varphi$ iff $\forall v \in W: wR_a v$ implies $\mathcal{M}, v \models \varphi$;
- $\mathcal{M}, w \models C_A \varphi$ iff $\forall v \in W: wR_a^+ v$ implies $\mathcal{M}, v \models \varphi$,

where $R_a^+$ is the transitive closure of $\bigcup_{a \in A} R_a$. 

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\[ \mathcal{M}, w \models K_a \varphi \quad \text{iff} \quad \forall v \in W: wR_a v \text{ implies } \mathcal{M}, v \models \varphi; \]
\[ \mathcal{M}, w \models C_A \varphi \quad \text{iff} \quad \forall v \in W: wR^+_A v \text{ implies } \mathcal{M}, v \models \varphi, \]

where $R^+_A$ is the transitive closure of $\bigcup_{a \in A} R_a$.

In other words, $C_A \varphi$ is true at $w$ iff any sequence of steps from $w$ along any agents’ relations ends in a state where $\varphi$ is true.
Example: Consecutive Numbers

Recall:

\[ M, w \models E\{b, g\} \land \neg g_5 \land \neg E\{b, g\} \land \neg g_5. \]

Pattern:

\[ M, w \models E\{b_2, g_3\} \land \neg g_6 \land \neg E\{b, g\} \land \neg g_6. \]
Basic Epistemic Logic

Example: Consecutive Numbers

Recall: $\mathcal{M}, w \models E_{\{b, g\}} \neg g_5 \land \neg E_{\{b, g\}} E_{\{b, g\}} \neg g_5$.
Example: Consecutive Numbers

Recall: $\mathcal{M}, w \models E_{\{b,g\}} \neg g_5 \land \neg E_{\{b,g\}} E_{\{b,g\}} \neg g_5$.

Pattern: $\mathcal{M}, w \models E_{\{b,g\}}^3 \neg g_6 \land \neg E_{\{b,g\}}^4 \neg g_6$. 
Example: Consecutive Numbers

Surprising result: \( M, w \models K_b g_3 \land K_g b_2 \land \neg C_{\{b, g\}} \neg b_{1000} \).

It’s not common knowledge that \( b \) doesn’t have 1000 on his head!
By contrast, in our earlier example with Berkeley and Copenhagen, it is common knowledge that if \( c \) is on vacation, then he doesn’t know whether or not it’s raining in Copenhagen:

\[
\mathcal{M}, w_1 \models C_{\{b, c\}} (v \rightarrow \neg (K_c r \lor K_c \neg r)).
\]
Idea 2: Common Knowledge as Transitive Closure

Definition
Given a $\kappa$-agent model $\mathcal{M} = \langle W, \{R_a\}_{a \in \text{Agt}}, V \rangle$ with $w \in W$ and $\varphi \in \mathcal{L}_{\text{EL-C}}^\kappa$, we define $\mathcal{M}, w \vDash \varphi$ as follows (other cases as before):

$$\mathcal{M}, w \vDash CA\varphi \iff \forall v \in W: wR_A^+ v \text{ implies } \mathcal{M}, v \vDash \varphi,$$

where $R_A^+$ is the transitive closure of $\bigcup_{a \in G} R_a$.

Observation
Given the infinitary character of common knowledge, it’s no surprise that with $\mathcal{L}_{\text{EL-C}}$ we lose compactness:

$$\{ E_A^n p \mid n \in \mathbb{N} \} \cup \{ \neg CAp \}$$

is finitely-satisfiable but not satisfiable.
Theorem

The $\mathcal{L}_{EL-C}$-theory of the class of all (resp. reflexive, preorder, partition, etc.) models is axiomatized by $K_\kappa$ (resp. $T_\kappa$, $S4_\kappa$, $S5_\kappa$, etc.) plus the following schemas for all $A \subseteq \text{Agt}$:

- **Fixed-Point Axiom:** $C_A \phi \leftrightarrow (\phi \land E_A C_A \phi)$

Complexity of checking satisfiability goes to EXPTIME-complete with common knowledge.
Completeness and Complexity

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The $\mathcal{L}_{EL}$-theory of the class of all (resp. reflexive, preorder, partition, etc.) models is axiomatized by $K_\kappa$ (resp. $T_\kappa$, $S4_\kappa$, $S5_\kappa$, etc.) plus the following schemas for all $A \subseteq \text{Agt}$:

- **Fixed-Point Axiom**: $C_A \varphi \leftrightarrow (\varphi \land E_A C_A \varphi)$

- **Induction Axiom**: $(\varphi \land C_A (\varphi \rightarrow E_A \varphi)) \rightarrow C_A \varphi$

Complexity of checking satisfiability goes to \textsc{EXPTIME}-complete with common knowledge.

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Completeness and Complexity

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- **Induction Axiom:** $(\varphi \land C_A (\varphi \to E_A \varphi)) \to C_A \varphi$

Complexity of checking satisfiability goes to EXPTIME-complete with common knowledge.
Idea 3: Information Update as Model Restriction

Example (Muddle Children Puzzle)

Three children—r, g, and b—have returned after playing outside. Their mother says to them, “At least one of you has mud on your forehead. Step forward now if you know whether you are dirty.”
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In fact, \(g\) and \(b\) are dirty, but no one can see his own forehead.
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None of the children step forward. So their mother says again, “Step forward now if you know whether you are dirty.”
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None of the children step forward. So their mother says again, “Step forward now if you know whether you are dirty.”

This time $g$ and $b$ both step forward, but $r$ does not.
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In fact, $g$ and $b$ are dirty, but no one can see his own forehead.

None of the children step forward. So their mother says again, “Step forward now if you know whether you are dirty.”

This time $g$ and $b$ both step forward, but $r$ does not.

So their mother says again. “Step forward now if you know whether you are dirty.” Finally $r$ steps forward.
“At least one of you is dirty.” $D_r \lor D_g \lor D_b$. What happens?
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“At least one of you is dirty.” $D_r \lor D_g \lor D_b$. What happens?
“If you know whether you’re dirty, step forward.”
“If you know whether you’re dirty, step forward.” No one does.
\[
\neg (K_r D_r \lor K_r \neg D_r) \land \neg (K_g D_g \lor K_g \neg D_g) \land \neg (K_b D_b \lor K_b \neg D_b).
\]
What happens after this is announced?
\[ \neg (K_r D_r \lor K_r \neg D_r) \land \neg (K_g D_g \lor K_g \neg D_g) \land \neg (K_b D_b \lor K_b \neg D_b) \].

What happens after this? Three more possibilities are eliminated.
“If you know whether you’re dirty, step forward.”
“If you know whether you’re dirty, step forward.” Blue and green (but not red) step forward.
“If you know whether you’re dirty, step forward.” Blue and green (but not red) step forward. \((K_b D_b \lor K_b \neg D_b) \land (K_g D_g \lor K_g \neg D_g)\).
Blue and green (but not red) step forward. \((K_b D_b \lor K_b \neg D_b) \land (K_g D_g \lor K_g \neg D_g)\). What happens after this is announced?
\[(K_b D_b \lor K_b \neg D_b) \land (K_g D_g \lor K_g \neg D_g)\). What happens after this? The remaining non-actual possibilities are eliminated.
Basic Epistemic Logic

\[
(CDD) \land (KbDb \lor Kb\neg Db) \land (KgDg \lor Kg\neg Dg).
\]

What happens after this? The remaining non-actual possibilities are eliminated.
One of the big ideas of **dynamic** epistemic logic is to add to our formal language operators that can describe the kinds of model changes that we just saw for the Muddy Children puzzle.
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**Definition (Basic Epistemic Languages)**

Given sets $\text{At} = \{p, q, r, \ldots \}$ and $\text{Agt} = \{a, b, c, \ldots \}$ with $|\text{Agt}| = \kappa$, the language of public announcement logic, $\mathcal{L}_{\text{PAL}}^\kappa$, is generated by

$$\varphi ::= p \mid \neg \varphi \mid (\varphi \land \varphi) \mid K_a \varphi \mid [\varphi] \varphi.$$
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Given sets $A_t = \{p, q, r, \ldots \}$ and $A_g = \{a, b, c, \ldots \}$ with $|A_g| = \kappa$, the language of public announcement logic, $L^\kappa_{\text{PAL}}$, is generated by

$$
\phi ::= p \mid \neg \phi \mid (\phi \land \phi) \mid K_a \phi \mid [\phi] \phi.
$$

Read $[\phi] \psi$ as “after (every) true announcement of $\phi$, $\psi$.”
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Read $\langle\varphi\rangle\psi := \neg[\varphi]\neg\psi$ as “after a true announcement of $\varphi$, $\psi$.”
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One can also add common knowledge $C_A$ to the language.
Definition (Basic Epistemic Languages)

Given sets $A_{t} = \{p, q, r, \ldots \}$ and $A_{t} = \{a, b, c, \ldots \}$ with $|A_{t}| = \kappa$, the epistemic language $\mathcal{L}_{EL}^{\kappa}$ is generated by

$p \mid \neg \varphi \mid (\varphi \land \varphi) \mid K_{a} \varphi.$

For the epistemic language with common knowledge, $\mathcal{L}_{EL-C}^{\kappa}$,

$p \mid \neg \varphi \mid (\varphi \land \varphi) \mid K_{a} \varphi \mid C \varphi.$

For the language of public announcement logic, $\mathcal{L}_{PAL}^{\kappa}$,

$p \mid \neg \varphi \mid (\varphi \land \varphi) \mid K_{a} \varphi \mid [\varphi] \varphi.$

For public announcement with common knowledge, $\mathcal{L}_{PAL-C}^{\kappa}$,

$p \mid \neg \varphi \mid (\varphi \land \varphi) \mid K_{a} \varphi \mid [\varphi] \varphi \mid C \varphi.$
Basic Epistemic Logic

Idea 3: Information Update as Model Restriction

Definition (Truth cont.)

Given a $\kappa$-agent model $\mathcal{M} = \langle W, \{R_a\}_{a \in \text{Agt}}, V \rangle$ with $w \in W$ and $\varphi \in \mathcal{L}^\kappa_{\text{PAL}}$, we define $\mathcal{M}, w \models \varphi$ as follows (other cases as before):

$$\mathcal{M}, w \models [\varphi]\psi \iff \mathcal{M}, w \models \varphi \text{ implies } \mathcal{M}_{|\varphi}, w \models \psi,$$

where $\mathcal{M}_{|\varphi} = \langle W_{|\varphi}, \{R_a_{|\varphi}\}_{a \in \text{Agt}}, V_{|\varphi} \rangle$ is the submodel such that

- $W_{|\varphi} = \{v \in W \mid \mathcal{M}, v \models \varphi\}$;
- $\forall a \in \text{Agt}: R_{a_{|\varphi}} = R_a \cap (W_{|\varphi} \times W_{|\varphi})$;
- $\forall p \in \text{At}: V_{|\varphi}(p) = V(p) \cap W_{|\varphi}$.
Basic Epistemic Logic

Public Announcement Logic

Read $[\varphi] \psi$ as “after (every) true announcement of $\varphi$, $\psi$.”

Read $\langle \varphi \rangle \psi := \neg [\varphi] \neg \psi$ as “after a true announcement of $\varphi$, $\psi$.”

Let’s compare. The truth clause for the dynamic operator $[\varphi]$ is:

$\mathcal{M}, w \models [\varphi] \psi$ iff $\mathcal{M}, w \models \varphi$ implies $\mathcal{M}_{|\varphi}, w \models \psi$. 

Big Idea: we evaluate $[\varphi] \psi$ and $\langle \varphi \rangle \psi$ not by looking at other worlds in the same model, but rather by looking at a new model.
Public Announcement Logic

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Let’s compare. The truth clause for the dynamic operator $[\varphi]$ is:

$\models M, w \models [\varphi] \psi$ iff $M, w \models \varphi$ implies $M|_{\varphi}, w \models \psi$.

So if $\varphi$ is false, $[\varphi] \psi$ is vacuously true.
Basic Epistemic Logic

Public Announcement Logic

Read \([\varphi]\psi\) as “after (every) true announcement of \(\varphi, \psi\).”

Read \(\langle\varphi\rangle\psi := \neg[\varphi]\neg\psi\) as “after a true announcement of \(\varphi, \psi\).”

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M, w \models [\varphi]\psi \text{ iff } M, w \models \varphi \text{ implies } M|_{\varphi}, w \models \psi.
\]

So if \(\varphi\) is false, \([\varphi]\psi\) is vacuously true. Here is the \(\langle\varphi\rangle\) clause:

\[
M, w \models \langle\varphi\rangle\psi \text{ iff } M, w \models \varphi \text{ and } M|_{\varphi}, w \models \psi.
\]
Public Announcement Logic

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Read $\langle \varphi \rangle \psi := \neg [\varphi] \neg \psi$ as “after a true announcement of $\varphi$, $\psi$.”

Let’s compare. The truth clause for the dynamic operator $[\varphi]$ is:

- $M, w \models [\varphi] \psi$ iff $M, w \models \varphi$ implies $M_{|\varphi}, w \models \psi$.

So if $\varphi$ is false, $[\varphi] \psi$ is vacuously true. Here is the $\langle \varphi \rangle$ clause:

- $M, w \models \langle \varphi \rangle \psi$ iff $M, w \models \varphi$ and $M_{|\varphi}, w \models \psi$.

**Big Idea:** we evaluate $[\varphi] \psi$ and $\langle \varphi \rangle \psi$ not by looking at other worlds in the same model, but rather by looking at a new model.
Where $A_1 := D_r \lor D_g \lor D_b$, check $\mathcal{M}$, $\text{CCD} \vDash \langle A_1 \rangle K_b D_b$. 
Where $A_1 := D_r \lor D_g \lor D_b$, check $\mathcal{M}$, $\text{C CD} \models \langle A_1 \rangle K_b D_b$. $A_1$ is Mom’s announcement: “At least one of you is dirty.”
Where \( A_1 := D_r \vee D_g \vee D_b \), check \( \mathcal{M}, CCD \models \langle A_1 \rangle K_b D_b \).

First, observe that \( \mathcal{M}, CCD \models D_r \vee D_g \vee D_b \).
Where $A_1 := D_r \lor D_g \lor D_b$, check $\mathcal{M}$, $\text{CCD} \vDash \langle A_1 \rangle K_b D_b$.

Second, delete all worlds where $D_r \lor D_g \lor D_b$ is false.
Basic Epistemic Logic

Where \( A_1 := D_r \lor D_g \lor D_b \), check \( \mathcal{M}, C C D \models \langle A_1 \rangle K_b D_b \).

Second, delete all worlds where \( D_r \lor D_g \lor D_b \) is false.
Basic Epistemic Logic

Where $A_1 := D_r \lor D_g \lor D_b$, check $\mathcal{M}, CCD \models \langle A_1 \rangle K_b D_b$.

Finally, observe that $\mathcal{M}_{|A_1}, CCD \models K_b D_b$. 
Where $A_2 := \bigwedge_{a \in \{r, g, b\}} \neg(K_a D_a \lor K_a \neg D_a)$, representing that no one steps forward, check $\mathcal{M}_{|A_1}, CDD \models \langle A_2 \rangle (K_g D_g \land K_b D_b)$. 
Basic Epistemic Logic

\[ M_{|A_1}, CDD \models \left\{ \wedge_{a \in \{r, g, b\}} \neg (K_a D_a \lor K_a \neg D_a) \right\} (K_g D_g \land K_b D_b). \]

First, note \[ M_{|A_1}, CDD \models \wedge_{a \in \{r, g, b\}} \neg (K_a D_a \lor K_a \neg D_a). \]
Basic Epistemic Logic

\[ M_{A_1}, \text{CDD} \models \langle \bigwedge_{a \in \{r,g,b\}} \neg (K_a D_a \lor K_a \neg D_a) \rangle (K_g D_g \land K_b D_b) \].

Second, delete all worlds where \( \bigwedge_{a \in \{r,g,b\}} \neg (K_a D_a \lor K_a \neg D_a) \) is false.
\[ M_{|A_1}, CDD \models \langle \bigwedge_{a \in \{r, g, b\}} \neg (K_a D_a \lor K_a \neg D_a) \rangle (K_g D_g \land K_b D_b). \]

Second, delete all worlds where \( \bigwedge_{a \in \{r, g, b\}} \neg (K_a D_a \lor K_a \neg D_a) \) is false.
Basic Epistemic Logic

\[ \mathcal{M}_{|A_1}, CDD \models \langle \bigwedge_{a \in \{r, g, b\}} \neg (K_a D_a \lor K_a \neg D_a) \rangle (K_g D_g \land K_b D_b). \]

Finally, observe \[ \mathcal{M}_{|A_1|} \bigwedge_{a \in \{r, g, b\}} \neg (K_a D_a \lor K_a \neg D_a), CDD \models K_g D_g \land K_b D_b. \]
Check $\mathcal{M}_{|A_1|A_2}$, $\text{CDD} \models \langle \bigwedge_{a \in \{g, b\}} (K_a D_a \lor K_a \neg D_a) \rangle K_r \neg D_r$. 
Check $\mathcal{M}|_{A_1|A_2}$, $\mathcal{CDD} \models \left\langle \bigwedge_{a \in \{g,b\}} (K_a D_a \lor K_a \neg D_a) \right\rangle K_r \neg D_r$.

This represents the info. given when $g$ and $b$ step forward after Mom says *again*, “If you know whether you’re dirty, step forward.”
Check $\mathcal{M}_{A_1|A_2}$, $CDD \models \langle \bigwedge_{a \in \{g, b\}} (K_a D_a \lor K_a \neg D_a) \rangle K_r \neg D_r.$

First, observe $\mathcal{M}_{A_1|A_2}$, $CDD \models \bigwedge_{a \in \{g, b\}} (K_a D_a \lor K_a \neg D_a).$
Check $\mathcal{M}_{|A_1|A_2}$, $\text{CDD} \models \langle \bigwedge_{a \in \{g,b\}} (K_aD_a \lor K_a\neg D_a) \rangle K_r \neg D_r$.

Second, delete all worlds where $\bigwedge_{a \in \{g,b\}} (K_aD_a \lor K_a\neg D_a)$ is false.
Check $\mathcal{M}_{|A_1|A_2}$, $\mathcal{CDD} \models \langle \bigwedge_{a \in \{g,b\}} (K_a D_a \lor K_a \neg D_a) \rangle K_r \neg D_r$.

Second, delete all worlds where $\bigwedge_{a \in \{g,b\}} (K_a D_a \lor K_a \neg D_a)$ is false.
Check $\mathcal{M}_{A_1|A_2}$, $\text{CDD} \models \langle \land_{a \in \{g,b\}} (K_aD_a \lor K_a\neg D_a) \rangle K_r\neg D_r$.

Finally, observe $\mathcal{M}_{A_1|A_2}$, $\text{CDD} \models K_r\neg D_r$. 
Theorem (PAL Axiomatization (Plaza 1989))

The $\mathcal{L}_{\text{PAL}}^\kappa$-theory of the class of all (resp. reflexive, preorder, partition) models is axiomatized by $\mathbf{K}_\omega$ (resp. $\mathbf{T}_\omega$, $\mathbf{S4}_\omega$, $\mathbf{S5}_\omega$) plus:

i. (replacement) $\psi \leftrightarrow \chi$  
   $\varphi(\psi/p) \leftrightarrow \varphi(\chi/p)$

ii. (atomic reduction) $\langle \varphi \rangle p \leftrightarrow (\varphi \land p)$

iii. (negation reduction) $\langle \varphi \rangle \neg \psi \leftrightarrow (\varphi \land \neg \langle \varphi \rangle \psi)$

iv. (conjunction reduction) $\langle \varphi \rangle (\psi \land \chi) \leftrightarrow (\langle \varphi \rangle \psi \land \langle \varphi \rangle \chi)$

v. (diamond reduction) $\langle \varphi \rangle \hat{K}_a \psi \leftrightarrow (\varphi \land \hat{K}_a \langle \varphi \rangle \psi)$.

Complexity of satisfiability for PAL is the same as for EL.
With the background of basic epistemic logic in place, let’s turn to mathematical themes arising in recent work in epistemic logic:

- Preservation Theorems
- Uniform Substitution
- Relations to FOL
Preservation of Truth (in Inquiry)

**Definition (Preservation Under Submodels)**

A formula $\varphi$ is preserved under submodels iff for any pointed model $\mathcal{M}, w$ and $\mathcal{M}' \subseteq \mathcal{M}$: $\mathcal{M}, w \models \varphi \Rightarrow \mathcal{M}', w \models \varphi$. 
Definition (Preservation Under Submodels)

A formula $\varphi$ is preserved under submodels iff for any pointed model $\mathcal{M}, w$ and $\mathcal{M}' \subseteq \mathcal{M}$: $\mathcal{M}, w \models \varphi \Rightarrow \mathcal{M}', w \models \varphi$.

Given our epistemic understanding of the models, this kind of preservation takes on a new meaning: once true, $\varphi$ remains true henceforth in inquiry, no matter what new knowledge is acquired.
Definition (Preservation Under Submodels)
A formula \( \varphi \) is preserved under submodels iff for any pointed model \( \mathcal{M}, \mathcal{M}' \subseteq \mathcal{M} \): \( \mathcal{M}, w \models \varphi \Rightarrow \mathcal{M}', w \models \varphi \).

Definition (Universal Fragment)
The universal fragment of \( \mathcal{L}^\kappa_{\text{EL}} \) is generated by:

\[
\varphi ::= p \mid \neg p \mid (\varphi \land \varphi) \mid (\varphi \lor \varphi) \mid K_a \varphi
\]

for \( p \in \text{At} \) and \( a \in \text{Agt} \).
Definition (Preservation Under Submodels)
A formula $\phi$ is preserved under submodels iff for any pointed model $M, w$ and $M' \subseteq M$: $M, w \models \phi \Rightarrow M', w \models \phi$.

Definition (Universal Fragment)
The universal fragment of $\mathcal{L}^\kappa_{EL}$ is generated by:

$$\phi ::= p \mid \neg p \mid (\phi \land \phi) \mid (\phi \lor \phi) \mid K_a \phi$$

for $p \in \text{At}$ and $a \in \text{Agt}$.

Theorem (Modal Łoś-Tarski)
A formula $\phi \in \mathcal{L}^\kappa_{EL}$ is preserved under submodels iff it is equivalent to a formula in the universal fragment of $\mathcal{L}^\kappa_{EL}$.
Definition (Preservation Under Submodels)
A formula $\phi$ is preserved under submodels iff for any pointed model $\mathcal{M}, w$ and $\mathcal{M}' \subseteq \mathcal{M}$: $\mathcal{M}, w \models \phi \Rightarrow \mathcal{M}', w \models \phi$.

Definition (Universal Fragment)
The universal fragment of $\mathcal{L}_{\text{PAL-C}}^k$ is generated by:

$$
\phi ::= p \mid \neg p \mid (\phi \land \phi) \mid (\phi \lor \phi) \mid K_a \phi \mid C_A \phi \mid [\neg \phi] \phi
$$

for $p \in \text{At}$ and $a \in \text{Agt}$.

Theorem (van Ditmarsch, Kooi)
A formula $\phi \in \mathcal{L}_{\text{PAL-C}}^k$ is preserved under submodels if it is equivalent to a formula in the universal fragment of $\mathcal{L}_{\text{PAL-C}}^k$.

Open Problem 1: does 'only if' hold as well?
Definition (Preservation Under Submodels)
A formula $\varphi$ is preserved under submodels iff for any pointed model $M, w$ and $M' \subseteq M$: $M, w \models \varphi \Rightarrow M', w \models \varphi$.

Definition (Universal Fragment)
The universal fragment of $\mathcal{L}^\kappa_{\text{PAL-C}}$ is generated by:

$$
\varphi ::= p \mid \neg p \mid (\varphi \land \varphi) \mid (\varphi \lor \varphi) \mid K_a \varphi \mid C_A \varphi \mid [\neg \varphi] \varphi
$$

for $p \in \text{At}$ and $a \in \text{Agt}$.

Theorem (van Ditmarsch, Kooi)

A formula $\varphi \in \mathcal{L}^\kappa_{\text{PAL-C}}$ is preserved under submodels if it is equivalent to a formula in the universal fragment of $\mathcal{L}^\kappa_{\text{PAL-C}}$. 

Open Problem 1: does 'only if' hold as well?
Definition (Preservation Under Submodels)
A formula $\varphi$ is preserved under submodels iff for any pointed model $M$, $w$ and $M' \subseteq M$: $M, w \models \varphi \Rightarrow M', w \models \varphi$.

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The universal fragment of $L^\kappa_{\text{PAL-C}}$ is generated by:

$$\varphi ::= p \mid \neg p \mid (\varphi \land \varphi) \mid (\varphi \lor \varphi) \mid K_a \varphi \mid C_a \varphi \mid [\neg \varphi] \varphi$$

for $p \in \text{At}$ and $a \in \text{Agt}$.

Theorem (van Ditmarsch, Kooi)
A formula $\varphi \in L^\kappa_{\text{PAL-C}}$ is preserved under submodels if it is equivalent to a formula in the universal fragment of $L^\kappa_{\text{PAL-C}}$.

Open Problem 1: does ‘only if’ hold as well?
Definition (Preservation Under Submodels)

A formula $\varphi$ is preserved under submodels iff for any pointed model $M, w$ and $M' \subseteq M$: $M, w \models \varphi \Rightarrow M', w \models \varphi$.

**Open Problem 1**: Which formulas of $L_{EL-C}$ and $L_{PAL-C}$ (with common knowledge), syntactically characterized, are preserved?

(A place to start might be the proof of Łoś-Tarski for the $\mu$-calculus (D’Agostino, 1997), of which $L_{EL-C}$ is a fragment.)
Definition (Successful Formulas)

A formula $\varphi$ is successful iff for all pointed models $\mathcal{M}, w$,

$$\mathcal{M}, w \models \varphi \Rightarrow \mathcal{M}_{|\varphi}, w \models \varphi.$$
Definition (Successful Formulas)

A formula \( \varphi \) is successful iff for all pointed models \( \mathcal{M}, w \),

\[ \mathcal{M}, w \models \varphi \Rightarrow \mathcal{M}_{|\varphi}, w \models \varphi. \]

Successful formulas are preserved under “self-defined” submodels.
Definition (Successful Formulas)

A formula $\varphi$ is **successful** iff for all pointed models $\mathcal{M}, w$,

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Successful formulas are preserved under “self-defined” submodels.

Not all formulas are successful. Some formulas are **unsuccessful**.
Definition (Successful Formulas)

A formula $\varphi$ is **successful** iff for all pointed models $\mathcal{M}, w$,

$$\mathcal{M}, w \models \varphi \Rightarrow \mathcal{M}_{|\varphi}, w \models \varphi.$$  

Successful formulas are preserved under “self-defined” submodels. Not all formulas are successful. Some formulas are **unsuccessful**.

The simplest example: the Copenhagen agent $c$ calls the Berkeley agent $b$ on the phone and says “You don’t know it’s raining in Copenhagen, but it’s raining in Copenhagen” ($\neg K_b r \land r$).
Definition (Successful Formulas)

A formula $\varphi$ is successful iff for all pointed models $M, w$,

$$M, w \models \varphi \Rightarrow M\rvert_{\varphi}, w \models \varphi.$$ 

Successful formulas are preserved under “self-defined” submodels.
Not all formulas are successful. Some formulas are unsuccessful.

The simplest example: the Copenhagen agent $c$ calls the Berkeley agent $b$ on the phone and says “You don’t know it’s raining in Copenhagen, but it’s raining in Copenhagen” ($\neg K_b r \land r$). 

\[ b, c \xleftrightarrow{\quad} r \quad W_1 \]
Let’s go back to the first time that no students step forward. This is like their announcing, “We don’t know whether we are dirty.”
Preservation Theorems

\[ \mathcal{M}_{|A_1}, \ CDD \models \left\langle \land_{a \in \{r, g, b\}} \neg (K_a D_a \lor K_a \neg D_a) \right\rangle (K_g D_g \land K_b D_b) \].

This is like their saying “We don’t know whether we’re dirty.”
Preservation Theorems

First, note $\mathcal{M} \models CDD \models \bigwedge_{a \in \{r,g,b\}} \neg (K_a D_a \lor K_a \neg D_a)$.  

$\mathcal{M}_{|A_1}, CDD \models \langle \bigwedge_{a \in \{r,g,b\}} \neg (K_a D_a \lor K_a \neg D_a) \rangle (K_g D_g \land K_b D_b)$.  

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Preservation Theorems

\[ M_{A_1}, CDD \models \langle \bigwedge_{a \in \{r,g,b\}} \neg(K_a D_a \lor K_a \neg D_a) \rangle (K_g D_g \land K_b D_b). \]

Second, delete all worlds where \( \bigwedge_{a \in \{r,g,b\}} \neg(K_a D_a \lor K_a \neg D_a) \) is false.
Preservation Theorems

\[ M \models \neg(\forall a \in \{r, g, b\}, K_a D_a \lor K_a \neg D_a) \land (K_g D_g \land K_b D_b) \]

Second, delete all worlds where \( \forall a \in \{r, g, b\}, \neg(\forall \neg(\forall K_a D_a \lor K_a \neg D_a) \land (K_g D_g \land K_b D_b) \) is false.
\[ M|_{A_1}, \text{CD} \models \langle \bigwedge_{a \in \{r,g,b\}} \neg (K_a D_a \lor K_a \neg D_a) \rangle (K_g D_g \land K_b D_b). \]

Finally, observe \[ M|_{A_1} \models \bigwedge_{a \in \{r,g,b\}} \neg (K_a D_a \lor K_a \neg D_a), \text{CD} \models K_g D_g \land K_b D_b. \]
Finally, observe $\mathcal{M}_{|A_1|} \models \bigwedge_{a \in \{r, g, b\}} \neg (K_a D_a \lor K_a \neg D_a) \land CDD \models K_g D_g \land K_b D_b$.

Hence where $\varphi := \bigwedge_{a \in \{r, g, b\}} \neg (K_a D_a \lor K_a \neg D_a)$,

$$\mathcal{M}_{|A_1|}, CDD \models \langle \varphi \rangle \neg \varphi.$$
Finally, observe $\mathcal{M}_{|A_1|} \models \bigwedge_{a \in \{r,g,b\}} \neg (K_a D_a \lor K_a \neg D_a)$, $\mathcal{CDD} \models K_g D_g \land K_b D_b$.

Hence where $\varphi := \bigwedge_{a \in \{r,g,b\}} \neg (K_a D_a \lor K_a \neg D_a)$,

$\mathcal{M}_{|A_1|}, \mathcal{CDD} \models \langle \varphi \rangle \neg \varphi$.

The childrens’ true announcement that no one knows makes it false that no one knows! This is a so-called unsuccessful update.
Definition (Successful Formulas)
A formula $\varphi$ is successful iff for all pointed models $M, w$,

$$M, w \vDash \varphi \Rightarrow M|_\varphi, w \vDash \varphi.$$  

Successful formulas are preserved under “self-defined” submodels. Not all formulas are successful. Some formulas are unsuccessful.

Example (Unsuccessful Formulas)
We’ve seen two examples so far:

- $\neg K_b r \land r$
- $\land_{a \in \{r, g, b\}} \neg (K_a D_a \lor K_a \neg D_a)$
Definition (Successful Formulas)

A formula $\varphi$ is successful over a class of models $C$ iff for all pointed models $M, w$ with $M$ in $C$,

\[ M, w \models \varphi \implies M|_{\varphi}, w \models \varphi. \]

Question

Which formulas of $L_{EL}^{\kappa}$, syntactically characterized, are successful over the class of all models? The class of partition models?
Definition (Successful Formulas)
A formula $\varphi$ is successful over a class of models $C$ iff for all pointed models $M, w$ with $M$ in $C$,

$M, w \models \varphi \Rightarrow M_{|\varphi}, w \models \varphi$.

Question
Which formulas of $L^\kappa_{EL}$, syntactically characterized, are successful over the class of all models? The class of partition models?

Proposition (Complexity of the Success Problem)
The problem of deciding whether a formula $\varphi \in L^1_{EL}$ is successful over the class of partition models is co-NP complete.
Definition (Successful Formulas)

A formula $\varphi$ is successful over a class of models $C$ iff for all pointed models $M, w$ with $M$ in $C$,

$$M, w \models \varphi \Rightarrow M\models_{\varphi}, w \models \varphi.$$ 

Question

Which formulas of $L^k_{EL}$, syntactically characterized, are successful over the class of all models? The class of partition models?

Proposition (Complexity of the Success Problem)

The problem of deciding whether a formula $\varphi \in L^1_{EL}$ is successful over the class of partition models is co-NP complete.

So we cannot expect a very simple syntactic characterization.
Question
Which formulas of $\mathcal{L}^\kappa_{\text{EL}}$, syntactically characterized, are successful over the class of all models? The class of partition models?
Question

Which formulas of $L^\kappa_{EL}$, syntactically characterized, are successful over the class of all models? The class of partition models?

Q2 of van Benthem’s “Open Problems in Logical Dynamics.”
Question
Which formulas of $\mathcal{L}^\kappa_{EL}$, syntactically characterized, are successful over the class of all models? The class of partition models?

Q2 of van Benthem’s “Open Problems in Logical Dynamics.”

For the case $\kappa = 1$ with partition models, this was answered in:

Preservation Theorems

Question

Which formulas of $L^\kappa_{\text{EL}}$, syntactically characterized, are successful over the class of all models? The class of partition models?

Q2 of van Benthem’s “Open Problems in Logical Dynamics.”

For the case $\kappa = 1$ with partition models, this was answered in:


In essence, we show that all unsuccessful formulas involve variations on the Moorean theme of “you don’t know it, but $p$."

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Question
Which formulas of $L_{EL}^\kappa$, syntactically characterized, are successful over the class of all models? The class of partition models? Q2 of van Benthem’s “Open Problems in Logical Dynamics.”

For the case $\kappa = 1$ with partition models, this was answered in:


In essence, we show that all unsuccessful formulas involve variations on the Moorean theme of “you don’t know it, but $p$.”

Open Problem 2: answer the question for $\kappa > 1$ or for non-partition models (cf. Saraf and Sourabh for some leads).
Other Interesting Formula Classes

Definition (Self-Refuting)

A formula $\varphi$ is self-refuting iff for all pointed models $\mathcal{M}, w,$

$$\mathcal{M}, w \models \varphi \Rightarrow \mathcal{M}_{|\varphi}, w \not\models \varphi.$$
Definition (Self-Refuting)

A formula $\varphi$ is **self-refuting** iff for all pointed models $M, w$,

$$M, w \models \varphi \Rightarrow M_{\varphi}, w \not\models \varphi.$$ 

Definition (Eventually Self-Refuting)

Define $M_{n\varphi}$ recursively by $M_{0\varphi} = M$, $M_{n+1\varphi} = (M_{n\varphi})_{\varphi}$.

$\varphi$ is **eventually self-refuting** iff for all pointed models $M, w$:

$$M, w \models \varphi \Rightarrow \exists n : M_{n\varphi}, w \not\models \varphi.$$
Other Interesting Formula Classes

**Definition (Self-Refuting)**
A formula $\phi$ is **self-refuting** iff for all pointed models $M, w$,

$$M, w \models \phi \Rightarrow M|_\phi, w \not\models \phi.$$  

**Definition (Eventually Self-Refuting)**
Define $M|_{n\phi}$ recursively by $M|_0\phi = M$, $M|_{n+1}\phi = (M|_{n}\phi)|_{\phi}$.

$\phi$ is **eventually self-refuting** iff for all pointed models $M, w$:

$$M, w \models \phi \Rightarrow \exists n : M|_{n\phi}, w \not\models \phi.$$  

**Definition (Always Learnable)**
$\phi$ is **always learnable** (for $a$) iff for any pointed model $M, w$:

$$M, w \models \phi \Rightarrow \exists \psi : M|_{\psi}, w \models K_a\phi.$$
A striking feature of epistemic logic with operators for information update is that it is not closed under uniform substitution: e.g., $[p]p$ is valid, but $[\neg K_br \land r](\neg K_br \land r)$ is not valid, as we saw.
Definition (Substitution Core & Schematic Validity)

A substitution is any function $\sigma : \text{At} \rightarrow \mathcal{L}$; and we define $(\cdot)^\sigma : \mathcal{L} \rightarrow \mathcal{L}$ as the extension such that $(\varphi)^\sigma$ is obtained from $\varphi$ by replacing, for all $p \in \text{At}(\varphi)$, each occurrence of $p$ in $\varphi$ by $\sigma(p)$. 

The substitution core of logic $\mathcal{L}$ is the set

$$\{ \varphi \in \mathcal{L} | (\varphi)^\sigma \in \mathcal{L} \text{ for all substitutions } \sigma \}.$$ 

Formulas in the substitution core are schematically valid.
Definition (Substitution Core & Schematic Validity)

A substitution is any function $\sigma: \text{At} \to L$; and we define $(\cdot)^\sigma: L \to L$ as the extension such that $(\varphi)^\sigma$ is obtained from $\varphi$ by replacing, for all $p \in \text{At}(\varphi)$, each occurrence of $p$ in $\varphi$ by $\sigma(p)$.

The substitution core of logic $L$ is the set

$$\{ \varphi \in L_L \mid (\varphi)^\sigma \in L \text{ for all substitutions } \sigma \}.$$

Formulas in the substitution core are schematically valid.
Example (Valid but not Schematically Valid)
Where $C$ is the class of all models, formulas that are in $\text{Th}_{\mathcal{L}_{\text{PAL}}^\kappa}(C)$ but are not in the substitution core of $\text{Th}_{\mathcal{L}_{\text{PAL}}^\kappa}(C)$ include ($\kappa \geq 1$):

\[
\begin{align*}
[p]p & \quad K_a p \to [p]K_a p \\
[p]K_a p & \quad K_a p \to [p](p \to K_a p) \\
[p](p \to K_a p) & \quad K_a(p \to q) \to (\langle q \rangle K_a r \to \langle p \rangle K_a r) \\
[p \land \neg K_a p] \neg(p \land \neg K_a p) & \quad (\langle p \rangle K_a r \land \langle q \rangle K_a r) \to \langle p \lor q \rangle K_a r.
\end{align*}
\]
Example (Valid but not Schematically Valid)

Where $C$ is the class of all models, formulas that are in $\text{Th}_{\mathcal{L}_{\text{PAL}}^\kappa}(C)$ but are not in the substitution core of $\text{Th}_{\mathcal{L}_{\text{PAL}}^\kappa}(C)$ include ($\kappa \geq 1$):

\[
\begin{align*}
[p]p & \quad K_a p \rightarrow [p] K_a p \\
[p] K_a p & \quad K_a p \rightarrow [p] (p \rightarrow K_a p) \\
[p] (p \rightarrow K_a p) & \quad K_a (p \rightarrow q) \rightarrow (\langle q \rangle K_a r \rightarrow \langle p \rangle K_a r) \\
[p \land \neg K_a p] \neg (p \land \neg K_a p) & \quad (\langle p \rangle K_a r \land \langle q \rangle K_a r) \rightarrow \langle p \lor q \rangle K_a r.
\end{align*}
\]

So what are the uniform principles of information dynamics?
Question

Is the substitution core of PAL axiomatizable?

Q1 in van Benthem’s “Open Problems in Logical Dynamics.”
Question

*Is the substitution core of PAL axiomatizable?*

**Q1** in van Benthem’s “Open Problems in Logical Dynamics.”

The answer is given in:


**Theorem (Axiomatization)**

*For the class C of all (resp. reflexive, preorder, partition) ω-agent models, the substitution core of Th_{PAL}^C is axiomatizable.*
Question

Is the substitution core of PAL axiomatizable?

Q1 in van Benthem’s “Open Problems in Logical Dynamics.”

The answer is given in:

Wesley Holliday, Tomohiro Hoshi, and Thomas Icard. 2012.

Theorem (Axiomatization)

For the class C of all (resp. reflexive, preorder, partition) \(\omega\)-agent models, the substitution core of \(\text{Th}_{\text{PAL}}^\omega(C)\) is axiomatizable.

Upshot: we have a logic for reasoning about information change in which the atomic sentences \(p\) can be genuine propositional variables, standing in for any propositions, even epistemic ones.
The axiomatization is obtained by adding to the normal modal logic $K_\omega$ (or $T_\omega$, $S4_\omega$, $S5_\omega$ depending on the model class):

1. (Uniformity) $\phi(\phi)\sigma$ for any substitution $\sigma$
2. (Necessitation) $\phi[p]\phi$
3. (Extensionality) $\phi \leftrightarrow \psi \langle \phi \rangle p \leftrightarrow \langle \psi \rangle p$
4. (Distribution) $[p](q \rightarrow r) \rightarrow ([p]q \rightarrow [p]r)$
5. (P-Seriality) $p \rightarrow \langle p \rangle \top$
6. (Truthfulness) $\langle p \rangle \top \rightarrow p$
7. (Top-Reflexivity) $p \rightarrow \langle \top \rangle p$
8. (Functionality) $\langle p \rangle q \rightarrow [p]q$
9. (PA-Commutativity) $\langle p \rangle ^K a q \rightarrow ^K a \langle p \rangle q$
10. (AP-Commutativity) $^K a \langle p \rangle q \rightarrow [p] ^K a q$
11. (Composition) $\langle p \rangle \langle q \rangle r \leftrightarrow \langle \langle p \rangle q \rangle r$
The axiomatization is obtained by adding to the normal modal logic $K_\omega$ (or $T_\omega$, $S4_\omega$, $S5_\omega$ depending on the model class):

1. (uniformity) $\frac{\varphi}{(\varphi)^\sigma}$ for any substitution $\sigma$
The axiomatization is obtained by adding to the normal modal logic $K_\omega$ (or $T_\omega$, $S4_\omega$, $S5_\omega$ depending on the model class):

1. (uniformity) \[
\frac{\phi}{(\phi)\sigma} \quad \text{for any substitution } \sigma
\]

2. (necessitation) \[
\frac{\phi}{[p]\phi}
\]
The axiomatization is obtained by adding to the normal modal logic $K_\omega$ (or $T_\omega$, $S4_\omega$, $S5_\omega$ depending on the model class):

1. (uniformity) \[ \frac{\varphi}{(\varphi)\sigma} \] for any substitution $\sigma$
2. (necessitation) \[ \frac{\varphi}{[p]\varphi} \]
3. (extensionality) \[ \frac{\varphi \leftrightarrow \psi}{\langle \varphi \rangle p \leftrightarrow \langle \psi \rangle p} \]
Uniform Substitution

The axiomatization is obtained by adding to the normal modal logic $K_\omega$ (or $T_\omega$, $S4_\omega$, $S5_\omega$ depending on the model class):

1. (uniformity) \[
\frac{\varphi}{(\varphi)^\sigma}
\] for any substitution $\sigma$

2. (necessitation) \[
\frac{\varphi}{[p]\varphi}
\]

3. (extensionality) \[
\varphi \leftrightarrow \psi
\]
   \[
\frac{\langle \varphi \rangle p \leftrightarrow \langle \psi \rangle p}{\langle \varphi \rangle p \leftrightarrow \langle \psi \rangle p}
\]

4. (distribution) \[
[p](q \rightarrow r) \rightarrow ([p]q \rightarrow [p]r)
\]
The axiomatization is obtained by adding to the normal modal logic $\mathbf{K}_\omega$ (or $\mathbf{T}_\omega$, $\mathbf{S4}_\omega$, $\mathbf{S5}_\omega$ depending on the model class):

1. (uniformity) $\varphi \to (\varphi)^\sigma$ for any substitution $\sigma$
2. (necessitation) $\varphi \to [p] \varphi$
3. (extensionality) $\varphi \leftrightarrow \psi \to \langle \varphi \rangle p \leftrightarrow \langle \psi \rangle p$
4. (distribution) $[p](q \to r) \to ([p]q \to [p]r)$
5. ($p$-seriality) $p \to \langle p \rangle \top$
The axiomatization is obtained by adding to the normal modal logic $K_\omega$ (or $T_\omega$, $S4_\omega$, $S5_\omega$ depending on the model class):

1. (uniformity) \[ \frac{\varphi}{(\varphi)^\sigma} \] for any substitution $\sigma$
2. (necessitation) \[ \frac{\varphi}{[p]\varphi} \]
3. (extensionality) \[ \frac{\varphi \leftrightarrow \psi}{\langle \varphi \rangle p \leftrightarrow \langle \psi \rangle p} \]
4. (distribution) \[ [p](q \rightarrow r) \rightarrow ([p]q \rightarrow [p]r) \]
5. ($p$-seriality) \[ p \rightarrow \langle p \rangle \top \]
6. (truthfulness) \[ \langle p \rangle \top \rightarrow p \]
Uniform Substitution

The axiomatization is obtained by adding to the normal modal logic $K_{\omega}$ (or $T_{\omega}$, $S4_{\omega}$, $S5_{\omega}$ depending on the model class):

1. (uniformity) $\frac{\varphi}{(\varphi)^{\sigma}}$ for any substitution $\sigma$
2. (necessitation) $\frac{\varphi}{[p]\varphi}$
3. (extensionality) $\frac{\varphi \leftrightarrow \psi}{\langle \varphi \rangle p \leftrightarrow \langle \psi \rangle p}$
4. (distribution) $[p](q \to r) \to ([p]q \to [p]r)$
5. ($p$-seriality) $p \to \langle p \rangle \top$
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6. (truthfulness) \( \langle p \rangle T \rightarrow p \)
7. (\( T \)-reflexivity) \( p \rightarrow \langle T \rangle p \)
8. (functionality) \( \langle p \rangle q \rightarrow [p]q \)
9. (pa-commutativity) \( \langle p \rangle \hat{K}a \quad q \rightarrow \hat{K}a \langle p \rangle q \)
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\langle p \rangle \langle q \rangle r \leftrightarrow \langle \langle p \rangle q \rangle r
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For the class $C$ of all (resp. reflexive, preorder, partition) $\omega$-agent models, the substitution core of $\text{Th}_{\omega}^{\text{PAL}}(C)$ is axiomatizable.

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Wesley Holliday, Tomohiro Hoshi, and Thomas Icard. 2012.


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Open Problem 3: Does this extend to $Th_{L_{\text{PAL-}}C}^\omega(C)$? Is there a Uniform Logic of Information Dynamics & Common Knowledge?

(The obstacle is that given the failure of compactness, we must use a finite version of the Henkin-style canonical model construction to prove completeness; but the use of the standard Fisher-Ladner closure to do so proves difficult in this case...)
Transferring the Questions to FOL

In “Open Problems in Logical Dynamics,” van Benthem poses the questions we have been discussing for FOL as well.

Open Problem 4: Characterize the formulas $\varphi$ of FOL with exactly one free variable $x$ such that for any FO structure $\mathcal{A}$ and variable assignment $s$, if $\mathcal{A}, s \models \varphi$, then $\mathcal{A}_{\varphi}, s \models \varphi$, where $\mathcal{A}_{\varphi}$ is the substructure of $\mathcal{A}$ with domain $\{d \in |\mathcal{A}| \mid \mathcal{A}, s[x:=d] \models \varphi\}$. 
Relations to First-Order Logic

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**Open Problem 5:** Extend the language of FOL with relativization operators $(\cdot)^\varphi$ for all $\varphi$ with exactly one free variable $x$, such that

$$\mathcal{A}, s \models (\psi)^\varphi \iff \text{if } \forall y \in FV(\psi) \mathcal{A}, s[x:=s(y)] \models \varphi, \text{ then } \mathcal{A}_\varphi, s \models \psi,$$

where $FV(\psi)$ is the set of free variables in $\psi$.

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What is the complete logic of FOL with relativization?

Since this logic is not closed under uniform substitution (e.g., $Px \to (Px)^Qz$ is valid, but $(Px \land \exists y \neg Py) \to (Px \land \exists y \neg Py)^Qz$ is not), is the substitution core of this logic axiomatizable?
Translating EL into FOL

Can we transfer our results for modal logic to fragments of FOL?
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Definition (Propositional Epistemic Language)
Given sets $At = \{p, q, r, \ldots \}$ and $Agt = \{a, b, c, \ldots \}$ with $|Agt| = \kappa$, the epistemic language $\mathcal{L}^\kappa_{EL}$ is generated by

$$\varphi ::= p \mid \neg \varphi \mid (\varphi \land \varphi) \mid K_a \varphi.$$ 

Consider a first-order language that contains, for every $p \in At$ and $a \in Agt$, a one-place predicate $P$ and a two-place predicate $R_a$. 
Relations to First-Order Logic

Translating EL into FOL

Definition (Standard Translation)

Fix two variables $x$ and $y$ and define functions $ST_x$ and $ST_y$:

- $ST_x(p) = P_x$;
- $ST_x(\neg \varphi) = \neg ST_x(\varphi)$;
- $ST_x(\varphi \land \psi) = ST_x(\varphi) \land ST_x(\psi)$;
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- \( ST_x(K_a \varphi) = \forall y (R_a xy \rightarrow ST_y(\varphi)) \);
- and similarly for \( ST_y \).

Any modal model \( A \) can be viewed as a first-order structure \( \mathcal{A} \) such that for any \( \varphi \in \mathcal{L}_{EL}^\kappa \), variable \( x \), and assignment \( s \),

\[
A, w \models \varphi \iff \mathcal{A}, s[x:=w] \models ST_x(\varphi)
\]
Under the translation, the modal language lives in the two-variable fragment of FOL. It also lives in the “guarded” fragment of FOL, since all the quantifiers have guards. The satisfiability/validity problem for formulas in both of these fragment is decidable.
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Van Benthem’s Characterization Theorem gives a semantic characterization of the FOL formulas (with one free variable) that are equivalent to the standard translation of some modal formula: they are the FOL formulas that are invariant for bisimulation.
In the other direction, any formula $\varphi$ of Monadic First-Order Logic with only $x$ free (and not in the scope of any quantifier) can be translated into a modal formula $\varphi'$ such that for any MFO structure $\mathcal{A}$ and assignment $s$, when we view $\mathcal{A}$ as a single-agent epistemic partition model $\mathcal{A}$ (with $R_a$ the universal relation),

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Since we have solved the success problem and substitution core problem for single-agent epistemic logic over partition models, we can transfer these results to the modal-like fragment of MFOL.

But it is a long way from here to answers for full FOL.
Directions from Here

Going beyond the basic epistemic logic we have covered today:

- **Quantified** epistemic logic ($K_a \exists x Spy(x)$ vs. $\exists x K_a Spy(x)$);
- “Softer” belief revision instead of “hard” knowledge update;
- Modeling private communication, eavesdropping, etc.;
- Epistemic logic and probability (qualitative and quantitative);
- Philosophical applications of EL, mentioned at the beginning;
- and more...
Conclusion

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- and more...

Two sources for learning about recents research directions are:

- *European Summer School for Logic, Language, and Information*. 
Thank you!
A says to S and P: “I have picked $x, y \in \mathbb{N}$ such that $1 < x < y$ and $x + y \leq 100$. Shortly I will whisper their sum $s = x + y$ to S only, and their product $p = x \cdot y$ to P only.” A acts accordingly. The following conversation between S and P then takes place:

1. P: “I don’t know the numbers.”
2. S: “I knew that you didn’t know them.”
3. P: “Now I know the numbers.”
4. S: “Now I know them too.”

**Question:** what are $x$ and $y$? (There is a unique answer.)