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Schematic Validity in Dynamic Epistemic Logic: Decidability*

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Abstract. Unlike standard modal logics, many dynamic epistemic logics are not closed under uniform substitution. The classic example is Public Announcement Logic (PAL), an extension of epistemic logic based on the idea of information acquisition as elimination of possibilities. In this paper, we address the open question of whether the set of *schematic validities* of PAL, the set of formulas all of whose substitution instances are valid, is decidable. We obtain positive answers for multi-agent PAL, as well as its extension with relativized common knowledge, PAL-RC. The conceptual significance of substitution failure is also discussed.

Keywords: modal logic, dynamic epistemic logic, Public Announcement Logic, schematic validity, substitution core, decidability.

1 Introduction

The *schematic validities* of a logic are those formulas all of whose substitution instances are valid [3]. Typically the set of schematic validities of a logic, its *substitution core*, coincides with the set of validities, in which case the logic is closed under uniform substitution. However, many dynamic epistemic logics axiomatized using reduction axioms [8,1,4,16] are not substitution-closed.[4] The classic example is Public Announcement Logic (PAL) [17,10]. In this paper, we consider the schematic validity problem for PAL and its extension PAL-RC with relativized common knowledge [4]. We answer positively the open question [3,2,4] of whether the substitution cores of multi-agent PAL and PAL-RC are decidable. The conceptual significance of substitution failure is also discussed.

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[4] Dynamic epistemic logics are not the only modal logics to have been proposed that are not closed under substitution. Other examples include the modal logic of "pure provability" [6], Åqvist's two-dimensional modal logic as discussed by Segerberg [18], and an epistemic-doxastic logic proposed by Halpern [11]. For each of these logics there is an axiomatization in which non-schematically valid axioms appear.

1.1 Review of Public Announcement Logic

Let us briefly recall the details of PAL. The language $\mathcal{L}_{\mathsf{PAL}}$ is defined as follows, for a countable set At of atomic sentences and a finite set Agt of agent symbols:

$$\varphi \ ::= \ p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid K_i\varphi \mid \langle\varphi\rangle\varphi,$$

where $p \in \mathsf{At}$ and $i \in \mathsf{Agt}$. We denote the set of atoms in φ by $\mathsf{At}(\varphi)$ and define $[\varphi]\psi$ as $\neg\langle\varphi\rangle\neg\psi$. As in epistemic logic, we take $K_i\varphi$ to mean that agent i knows or has the information φ. For the "announcement" operator, we take $\langle\varphi\rangle\psi$ to mean that ψ is the case after all agents publicly receive the true information φ.

We interpret $\mathcal{L}_{\mathsf{PAL}}$ using standard relational structures of the form $\mathcal{M} = \langle M, \{\sim_i \mid i \in \mathsf{Agt}\}, V\rangle$, where each $\sim_i$ is an equivalence relation on M. We use the notation $\sim_i(w) = \{v \in M \mid w \sim_i v\}$ to denote the set of possibilities consistent with the knowledge or information of agent i in world w. Each K_i is the universal modality for the associated $\sim_i$ relation, and each $\langle\varphi\rangle$ is a dynamic modality corresponding to a model-relativization, with the following truth definitions:

$$\begin{array}{lll} \mathcal{M}, w \vDash K_i\varphi & \text{iff} & \forall v \in W\colon \text{if } w \sim_i v \text{ then } \mathcal{M}, v \vDash \varphi; \\ \mathcal{M}, w \vDash \langle\varphi\rangle\psi & \text{iff} & \mathcal{M}, w \vDash \varphi \text{ and } \mathcal{M}_{|\varphi}, w \vDash \psi, \end{array}$$

where $\mathcal{M}_{|\varphi} = \langle M_{|\varphi}, \{\sim_{i_{|\varphi}} \mid i \in \mathsf{Agt}\}, V_{|\varphi}\rangle$ is the model obtained by eliminating from $\mathcal{M}$ all worlds in which φ was false, i.e., $M_{|\varphi} = \{v \in M \mid \mathcal{M}, v \vDash \varphi\}$, each relation $\sim_{i_{|\varphi}}$ is the restriction of $\sim_i$ to $M_{|\varphi}$, and $V_{|\varphi}(p) = V(p) \cap M_{|\varphi}$ for all $p \in \mathsf{At}$. We denote the extension of φ in $\mathcal{M}$ by $[\![\varphi]\!]^{\mathcal{M}} = \{v \in M \mid \mathcal{M}, v \vDash \varphi\}$.

In essence, the semantics of PAL is based on the intuitive idea of *information acquisition as elimination of possibilities*, as illustrated by Example 2 below. An axiomatization of PAL is given by the **S5** axioms for each K_i modality, the rule of replacement of logical equivalents (from $\alpha \leftrightarrow \beta$, derive $\varphi(\alpha/p) \leftrightarrow \varphi(\beta/p)$), and the following reduction axioms [17]:

(i) $\langle\varphi\rangle p \leftrightarrow (\varphi \wedge p)$;
(ii) $\langle\varphi\rangle\neg\psi \leftrightarrow (\varphi \wedge \neg\langle\varphi\rangle\psi)$;
(iii) $\langle\varphi\rangle(\psi \wedge \chi) \leftrightarrow (\langle\varphi\rangle\psi \wedge \langle\varphi\rangle\chi)$;
(iv) $\langle\varphi\rangle K_i\psi \leftrightarrow (\varphi \wedge K_i(\varphi \to \langle\varphi\rangle\psi))$.

Using (i) - (iv) and replacement, any $\mathcal{L}_{\mathsf{PAL}}$ formula can be reduced to an equivalent formula in the basic modal language. Completeness and decidability for PAL are therefore corollaries of completeness and decidability for multi-agent **S5**.

The language of PAL-RC [4], $\mathcal{L}_{\mathsf{PAL\text{-}RC}}$, extends $\mathcal{L}_{\mathsf{PAL}}$ with *relativized common knowledge* operators $C^{\varphi}\psi$ with the truth definition: $\mathcal{M}, w \vDash C^{\varphi}\psi$ iff every path from w through $[\![\varphi]\!]^{\mathcal{M}}$ along any $\sim_i$ relations ends in $[\![\psi]\!]^{\mathcal{M}}$. The standard notion of common knowledge, that everyone knows ψ, and everyone knows that everyone knows that ψ, etc., is defined as $C\psi := C^{\top}\psi$. Using the reduction axiom

(v) $\langle\varphi\rangle C^{\psi}\chi \leftrightarrow (\varphi \wedge C^{\langle\varphi\rangle\psi}\langle\varphi\rangle\chi)$,

every $\mathcal{L}_{\mathsf{PAL\text{-}RC}}$ formula can be reduced to an equivalent formula without dynamic operators. Therefore, an axiomatization for PAL-RC may be obtained from (i) - (v) plus an axiomatization for multi-agent **S5** with relativized common knowledge [4]. Since the latter system is decidable, so is PAL-RC by the reduction.

1.2 Conceptual Significance of Substitution Failure

Reduction axiom (i) reflects an important assumption of PAL: the truth values of atomic sentences $p, q, r, \dots$ are taken to be unaffected by informational events. It is implicitly assumed that no atomic sentence is *about* the epistemic or informational states of agents. Hence an atomic sentence in PAL is not a *propositional variable* in the ordinary sense of something that stands in for any proposition. For there is an implicit restriction on the atomic sentence's subject matter.

Purists may protest that the atomic sentences of a real *logic* are supposed to be "topic-neutral." Our reply is practical: it is useful for certain applications to use the atomic $p, q, r, \dots$ to describe stable states of the external world, unaffected by informational events, while using modal formulas to describe the changeable states of agents' knowledge or information. As we show in other work [14], it is possible to develop a variant of PAL, which we call Uniform Public Announcement Logic (UPAL), in which atomic sentences are treated as genuine propositional variables. Which way one goes is a modeling choice.

Given the special treatment of atomic sentences in PAL, it is perhaps unsurprising that uniform substitution should fail. For example, the substitution instance $\langle p \rangle K_i p \leftrightarrow (p \wedge K_i p)$ of reduction axiom (i) is not valid. Since we take $K_i p$ to mean that agent i knows or has the information p, if $\langle p \rangle K_i p \leftrightarrow (p \wedge K_i p)$ were valid, it would mean that an agent could learn p only if the agent already knew p. Since PAL is designed to reason about information *change*, the non-schematic validity of reduction axiom (i) is a feature of the system, not a bug.

Although substitution failures are to be expected in PAL, the specific failures illuminate subtleties of information change. Example 1 provides the classic example. Example 2 shows that some substitution failures are not at all obvious.

Example 1 (Moore Sentence). The formula $[p]K_i p$ is valid, for when agent i acquires the information p, agent i comes to know p. Yet this formula is not schematically valid, and neither is the valid formula $[p]p$. Simply substitute the famous Moore sentence $p \wedge \neg K_i p$ for p. The non-schematic validity of $[p]p$ is the well-known issue of "unsuccessful formulas" [9,8,15], which is also at the heart of the Muddy Children puzzle [9, §4]. In these cases, the failure of schematic validity for a valid PAL principle shows that the principle does not hold for all types of information—in particular, for information about agents' own information.

Not only is the substitution instance $[p \wedge \neg K_i p](p \wedge \neg K_i p)$ of $[p]p$ invalid, but also $[p \wedge \neg K_i p]\neg(p \wedge \neg K_i p)$ is valid. Is the latter also schematically valid? Informally, is there a φ such that if you receive the true information that "φ but you don't know φ," it can *remain true* afterward that φ but you don't know φ? As Hintikka [12] remarks about sentences of the Moorean form, "If you know that I am well informed and if I address the words ... to you, these words have a curious effect which may perhaps be called anti-performatory. You may come to know that what I say *was* true, but saying it in so many words has the effect of making what is being said false" (p. 68f). Surprisingly, this is not always so.

Example 2 (Puzzle of the Gifts [13]). Holding her hands behind her back, agent i walks into a room where a friend j is sitting. Agent j did not see what if anything

i put in her hands, and i knows this. In fact, i has gifts for j in both hands. Instead of the usual game of asking j to "pick a hand, any hand," i (deviously but) truthfully announces:

(G) Either I have a gift in my *right* hand and you don't know that, or I have gifts in *both* hands and you don't know I have a gift in my *left* hand.

Let us suppose that j knows i to be an infallible source of information on such matters, so j accepts G. Question 1: After i's announcement, does j know whether i has a gift in her left/right/both hand(s)? Question 2: After i's announcement, is G *true*? Question 3: After i's announcement, does j *know* G? Finally, Question 4: If 'yes' to Q2, then what happens if i announces G again?

Let l stand for 'a gift is in i's left hand' and r stand for 'a gift is in i's right hand'. Before i's announcement, j has not eliminated any of the four possibilities represented by the model $\mathcal{M}$ in Fig. 1. (Reflexive arrows are not displayed.)

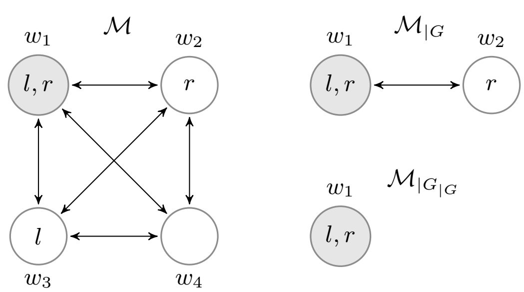

Fig. 1: models for the Puzzle of the Gifts

We can translate G into our language as

(*G*) $(r \wedge \neg K_j r) \vee (l \wedge r \wedge \neg K_j l)$.

Clearly $[\![G]\!]^{\mathcal{M}} = \{w_1, w_2\}$. Hence after i's announcement of G, j can eliminate possibilities w_3 and w_4, reducing j's uncertainty to that represented by the model $\mathcal{M}_{|G}$ in Fig. 1. Inspection of $\mathcal{M}_{|G}$ shows that the answer to Question 1 is that $\mathcal{M}_{|G}, w_1 \vDash K_j r \wedge \neg(K_j l \vee K_j \neg l)$. Observe that $[\![G]\!]^{\mathcal{M}_{|G}} = \{w_1\}$, which answers Questions 2 ('yes') and 3 ('no'). It follows that the principle $\langle\varphi\rangle\varphi \to \langle\varphi\rangle K_j\varphi$ is not schematically valid. One can fail to come to know what is (true *and remains true* after being) announced by a source whom one knows to be infallible!

Suppose that instead of initially announcing G, i announces

(*H*) $G \wedge \neg K_j G$.[5]

[5] "The following is true but you don't know it: either I have a gift in my *right* hand and you don't know that, or I have gifts in *both* hands and you don't know I have a gift in my *left* hand."

Given $[\![G]\!]^{\mathcal{M}} = \{w_1, w_2\}$ above, clearly $[\![H]\!]^{\mathcal{M}} = \{w_1, w_2\}$. It follows that $\mathcal{M}_{|G} = \mathcal{M}_{|H}$, so given $[\![G]\!]^{\mathcal{M}_{|G}} = \{w_1\}$ above, clearly $[\![H]\!]^{\mathcal{M}_{|H}} = \{w_1\}$. It follows that $\mathcal{M}, w_1 \vDash \langle H \rangle H$. Hence $[p \wedge \neg Kp]\neg(p \wedge \neg Kp)$ is valid but not schematically valid. Announcements of Moore sentences are not always self-refuting!

We leave the answer to Question 4 to the reader (see $\mathcal{M}_{|G_{|G}}$ in Fig. 1).

There are many other examples of valid but not schematically valid PAL principles. Noteworthy instances include $K_i(p \to q) \to (\langle q \rangle K_i r \to \langle p \rangle K_i r)$ and $(\langle p \rangle K_i r \wedge \langle q \rangle K_i r) \to \langle p \vee q \rangle K_i r$. Example 2 shows that discovering that there is an invalid substitution instance of a valid PAL formula can be a non-trivial task. A natural question is whether we can give an effective procedure to make such discoveries. The rest of the paper addresses this technical question.

1.3 The Problem of the Substitution Core

Let us now state precisely the problem to be solved. For a language $\mathcal{L}$ whose set of atomic sentences is At, a substitution is any function $\sigma : \mathsf{At} \to \mathcal{L}$, and $\hat{\sigma} : \mathcal{L} \to \mathcal{L}$ is the extension such that $\hat{\sigma}(\varphi)$ is obtained from φ by replacing each $p \in \mathsf{At}(\varphi)$ by $\sigma(p)$. Abusing notation, we write $\sigma(\varphi)$ for $\hat{\sigma}(\varphi)$. A formula φ is *schematically valid* iff for all such σ, $\sigma(\varphi)$ is valid. The substitution core of PAL is the set $\{\varphi \in \mathcal{L}_{\mathsf{PAL}} \mid \varphi \text{ schematically valid}\}$ and similarly for PAL-RC.

In van Benthem's list of "Open Problems in Logical Dynamics" [3], Question 1 is whether the substitution core of PAL-RC is decidable. We answer this question positively for PAL and PAL-RC in the following section.

2 Decidability

The idea of our proof is to provide a procedure for constructing a finite set of substitution instances for a given formula φ, such that if φ is not schematically valid, then there is a falsifiable substitution instance in the finite set. Suppose that for some substitution σ and model $\mathcal{M}$, we have $\mathcal{M}, w \nvDash \sigma(\varphi)$. From σ and $\mathcal{M}$, we will construct a special substitution τ such that $\tau(\varphi)$ is false at w in a suitable extension (on the valuation function) of $\mathcal{M}$. The construction reveals that τ is in a finite set of substitutions determined solely by the structure of φ. Therefore, to check whether φ is schematically valid, we need only check the validity of finitely many substitution instances of φ, which is a decidable problem for PAL and PAL-RC. We begin with a preliminary definition and result.

Definition 1. The set of *simple* formulas is defined as the smallest set such that: all $p \in \mathsf{At}$ are simple; if φ is simple, so are $\neg\varphi$, $K_i\varphi$, and $\langle \varphi \rangle \pm p$, where $\pm p$ is either p or $\neg p$ for $p \in \mathsf{At}$; if φ and ψ are simple, so are $\varphi \wedge \psi$ and $C^{\varphi}\psi$.

Proposition 1. For every formula $\varphi \in \mathcal{L}_{\mathsf{PAL\text{-}RC}}$, there is an equivalent simple formula φ'.

Proof By induction on φ, using the schematic validities (ii) - (v) in §1 and the schematic validity $\langle p \rangle \langle q \rangle r \leftrightarrow \langle \langle p \rangle q \rangle r$ [7]. □

2.1 Transforming Substitutions

Fix a formula φ in $\mathcal{L}_{\mathsf{PAL}}$ or $\mathcal{L}_{\mathsf{PAL\text{-}RC}}$. By Proposition 1, we may assume that φ is simple. Suppose that for some substitution σ and $\mathcal{M} = \langle M, \{\sim_i \mid i \in \mathsf{Agt}\}, V\rangle$, we have $\mathcal{M}, w \nvDash \sigma(\varphi)$. We will now provide a procedure to construct a special substitution τ from σ and a model $\mathcal{N}$ from $\mathcal{M}$, as discussed above, such that $\mathcal{N}, w \nvDash \tau(\varphi)$. Whether φ is in $\mathcal{L}_{\mathsf{PAL}}$ or $\mathcal{L}_{\mathsf{PAL\text{-}RC}}$, the resulting formula $\tau(\varphi)$ will be in $\mathcal{L}_{\mathsf{PAL\text{-}RC}}$. However, in §2.2 we will obtain substitution instances in $\mathcal{L}_{\mathsf{PAL}}$.

To construct $\tau(p)$ for a given $p \in \mathsf{At}$, let $B_1, \dots, B_m$ be the sequence of all B_i such that $\langle B_i\rangle \pm p$ occurs in φ, and let $B_0 := \top$. For $0 \leq i, j \leq m$, if $[\![\sigma(B_i)]\!]^{\mathcal{M}} = [\![\sigma(B_j)]\!]^{\mathcal{M}}$, then delete one of B_i or B_j from the list (but never B_0), until there is no such pair. Call the resulting sequence $A_0, \dots, A_n$, and define

$$s(i) = \{j \mid 0 \leq j \leq n \text{ and } [\![\sigma(A_j)]\!]^{\mathcal{M}} \subset [\![\sigma(A_i)]\!]^{\mathcal{M}}\}.$$

Extend the language with new variables $p_0, \dots, p_n$ and $a_0, \dots, a_n$, and define $\tau(p) = \kappa_1 \wedge \dots \wedge \kappa_n$ such that

$$\kappa_i := p_i \vee \bigvee_{0 \leq j \leq n,\, j \neq i} \Big(Ca_j \wedge \bigwedge_{0 \leq k \leq n,\ k \in s(j)} \neg Ca_k \Big).$$

Without loss of generality, we assume that $\mathcal{M}$ is *generated* by $\{w\}$ [5, Def. 2.5], so the C operator in κ_i functions as the global modality in $\mathcal{M}$.

Having extended the language for each $p \in \mathsf{At}(\varphi)$, extend the valuation V to V' such that for each $p \in \mathsf{At}(\varphi)$, $V'(p) = V(p)$, and for the new variables:

(a) $V'(p_i) = [\![\sigma(p)]\!]^{\mathcal{M}|_{\sigma(A_i)}}$;
(b) $V'(a_i) = [\![\sigma(A_i)]\!]^{\mathcal{M}}$.

Let $\mathcal{N} = \langle M, \{\sim_i \mid i \in \mathsf{Agt}\}, V'\rangle$ be the extension of $\mathcal{M}$ with the new V'.

We will show that $\tau(p)$ has the same extension as $\sigma(p)$ after relativization by any $\sigma(A_i)$, which has the same extension as $\tau(A_i)$. It will follow that $\mathcal{N}, w \nvDash \tau(\varphi)$ given $\mathcal{M}, w \nvDash \sigma(\varphi)$.

Fact 1 For $p \in \mathsf{At}(\varphi)$, $[\![\sigma(\langle A_i\rangle \pm p)]\!]^{\mathcal{M}} = [\![\pm p_i]\!]^{\mathcal{N}}$.

Proof By basic definitions,

$$\begin{aligned} [\![\sigma(\langle A_i\rangle \pm p)]\!]^{\mathcal{M}} &= [\![\langle \sigma(A_i)\rangle \pm \sigma(p)]\!]^{\mathcal{M}} \\ &= [\![\pm\sigma(p)]\!]^{\mathcal{M}|_{\sigma(A_i)}} \\ &= [\![\pm p_i]\!]^{\mathcal{N}}, \end{aligned}$$

where the last equality holds by (a) and the definition of $\mathcal{N}$. □

Lemma 1. For $p \in \mathsf{At}(\varphi)$ and $0 \le i \le n$,

$$[\![\tau(p)]\!]^{\mathcal{N}_{|a_i}} = [\![p_i]\!]^{\mathcal{N}}.$$

Proof We first show that for $0 \le i, j \le n$, $i \ne j$:

1. $[\![\kappa_i]\!]^{\mathcal{N}_{|a_i}} = [\![p_i]\!]^{\mathcal{N}_{|a_i}}$;
2. $[\![\kappa_j]\!]^{\mathcal{N}_{|a_i}} = [\![a_i]\!]^{\mathcal{N}_{|a_i}}$ $(= M_{|a_i})$.

For 1, we claim that given $i \ne j$,

$$[\![Ca_j \wedge \bigwedge_{0 \le k \le n,\ k \in s(j)} \neg Ca_k]\!]^{\mathcal{N}_{|a_i}} = \emptyset.$$

By construction of the sequence $A_0, \ldots, A_n$ for p and (b), $[\![a_j]\!]^{\mathcal{N}} \ne [\![a_i]\!]^{\mathcal{N}}$. If $[\![a_i]\!]^{\mathcal{N}} \not\subset [\![a_j]\!]^{\mathcal{N}}$, then $[\![Ca_j]\!]^{\mathcal{N}_{|a_i}} = \emptyset$. If $[\![a_i]\!]^{\mathcal{N}} \subset [\![a_j]\!]^{\mathcal{N}}$, then by (b) and the definition of s, $i \in s(j)$. Then since a_i is propositional, $[\![\neg Ca_i]\!]^{\mathcal{N}_{|a_i}} = \emptyset$. In either case the claim holds, so $[\![\kappa_i]\!]^{\mathcal{N}_{|a_i}} = [\![p_i]\!]^{\mathcal{N}_{|a_i}}$ given the structure of κ_i.

For 2, κ_j contains as a disjunct:

$$Ca_i \wedge \bigwedge_{0 \le k \le n,\ k \in s(i)} \neg Ca_k.$$

Since a_i is propositional, $[\![Ca_i]\!]^{\mathcal{N}_{|a_i}} = M_{|a_i}$. By definition of s and (b), for all $k \in s(i)$, $[\![a_k]\!]^{\mathcal{N}} \subset [\![a_i]\!]^{\mathcal{N}}$, which gives $[\![\neg Ca_k]\!]^{\mathcal{N}_{|a_i}} = M_{|a_i}$. Hence $[\![\kappa_j]\!]^{\mathcal{N}_{|a_i}} = M_{|a_i}$.

Given the construction of τ, 1 and 2 imply:

$$[\![\tau(p)]\!]^{\mathcal{N}_{|a_i}} = [\![\kappa_i]\!]^{\mathcal{N}_{|a_i}} \cap \bigcap_{j \ne i} [\![\kappa_j]\!]^{\mathcal{N}_{|a_i}} = [\![p_i]\!]^{\mathcal{N}_{|a_i}} \cap [\![a_i]\!]^{\mathcal{N}_{|a_i}} = [\![p_i]\!]^{\mathcal{N}}.$$

The last equality holds because $[\![p_i]\!]^{\mathcal{N}} \subseteq [\![a_i]\!]^{\mathcal{N}}$, which follows from (a) and (b). □

Lemma 2. For all simple subformulas χ of φ,

$$[\![\tau(\chi)]\!]^{\mathcal{N}} = [\![\sigma(\chi)]\!]^{\mathcal{M}}.$$

Proof By induction on χ. For the base case, we must show $[\![\tau(p)]\!]^{\mathcal{N}} = [\![\sigma(p)]\!]^{\mathcal{M}}$. By construction of the sequence $A_0, \ldots, A_n$ for $p \in \mathsf{At}(\varphi)$, there is some $A_j = \top$, so $[\![\sigma(A_j)]\!]^{\mathcal{M}} = M$. Then by (b), $[\![a_j]\!]^{\mathcal{N}} = M$, and hence

$$\begin{aligned} [\![\tau(p)]\!]^{\mathcal{N}} &= [\![\tau(p)]\!]^{\mathcal{N}_{|a_j}} \\ &= [\![p_j]\!]^{\mathcal{N}} && \text{by Lemma 1} \\ &= [\![\sigma(p)]\!]^{\mathcal{M}_{|\sigma(A_j)}} && \text{by } (a) \\ &= [\![\sigma(p)]\!]^{\mathcal{M}}. \end{aligned}$$

The boolean cases are straightforward. Next, we must show $[\![\tau(K_k\varphi)]\!]^{\mathcal{N}} = [\![\sigma(K_k\varphi)]\!]^{\mathcal{M}}$. For the inductive hypothesis, we have $[\![\tau(\varphi)]\!]^{\mathcal{N}} = [\![\sigma(\varphi)]\!]^{\mathcal{M}}$, so

$$\begin{aligned} [\![\tau(K_k\varphi)]\!]^{\mathcal{N}} &= [\![K_k\tau(\varphi)]\!]^{\mathcal{N}} \\ &= \{w \in M \mid \sim_k(w) \subseteq [\![\tau(\varphi)]\!]^{\mathcal{N}}\} \\ &= \{w \in M \mid \sim_k(w) \subseteq [\![\sigma(\varphi)]\!]^{\mathcal{M}}\} \\ &= [\![K_k\sigma(\varphi)]\!]^{\mathcal{M}} \\ &= [\![\sigma(K_k\varphi)]\!]^{\mathcal{M}}. \end{aligned}$$

Similar reasoning applies in the case of $C^{\varphi}\psi$.

Finally, we must show $[\![\tau(\langle B_i\rangle \pm p)]\!]^{\mathcal{N}} = [\![\sigma(\langle B_i\rangle \pm p)]\!]^{\mathcal{M}}$. For the inductive hypothesis, $[\![\tau(B_i)]\!]^{\mathcal{N}} = [\![\sigma(B_i)]\!]^{\mathcal{M}}$. By construction of the sequence $A_0, \dots, A_n$ for $p \in \mathsf{At}(\varphi)$, there is some A_j such that $[\![\sigma(B_i)]\!]^{\mathcal{M}} = [\![\sigma(A_j)]\!]^{\mathcal{M}}$. Therefore,

$$\begin{aligned} [\![\tau(B_i)]\!]^{\mathcal{N}} &= [\![\sigma(A_j)]\!]^{\mathcal{M}} \\ &= [\![a_j]\!]^{\mathcal{N}} && \text{by (b),} \end{aligned}$$

and hence

$$\begin{aligned} [\![\tau(\langle B_i\rangle \pm p)]\!]^{\mathcal{N}} &= [\![\langle \tau(B_i)\rangle \pm \tau(p)]\!]^{\mathcal{N}} \\ &= [\![\langle a_j\rangle \pm \tau(p)]\!]^{\mathcal{N}} \\ &= [\![\pm\tau(p)]\!]^{\mathcal{N}|_{a_j}} \\ &= [\![\pm p_j]\!]^{\mathcal{N}} && \text{by Lemma 1} \\ &= [\![\sigma(\langle A_j\rangle \pm p)]\!]^{\mathcal{M}} && \text{by (a)} \\ &= [\![\sigma(\langle B_i\rangle \pm p)]\!]^{\mathcal{M}} && \text{given } [\![\sigma(B_i)]\!]^{\mathcal{M}} = [\![\sigma(A_j)]\!]^{\mathcal{M}}. \end{aligned}$$

The proof by induction is complete. □

Fact 2 $\mathcal{N}, w \nvDash \tau(\varphi)$.

Proof Immediate from Lemma 2 given $\mathcal{M}, w \nvDash \sigma(\varphi)$. □

2.2 Proof of Decidability

Given $\mathcal{M}, w \nvDash \sigma(\varphi)$, using the procedure of §2.1, we can construct a special substitution τ and an extended model $\mathcal{N}$ with $\mathcal{N}, w \nvDash \tau(\varphi)$. It is clear from the procedure that we need $\mathcal{M}$, σ, and φ to construct τ. For each $p \in \mathsf{At}(\varphi)$, given the subformulas $A_0, \dots, A_n$ of φ, we defined $\tau(p) = \kappa_1 \wedge \dots \wedge \kappa_n$, where

$$\kappa_i := p_i \vee \bigvee_{0 \le j \le n,\, j \ne i} \Big(Ca_j \wedge \bigwedge_{0 \le k \le n,\ k \in s(j)} \neg Ca_k \Big).$$

Since we defined $s(i) = \{ j \mid 0 \le j \le n \text{ and } [\![\sigma(A_j)]\!]^{\mathcal{M}} \subset [\![\sigma(A_i)]\!]^{\mathcal{M}} \}$ for $i \le n$, we required information from σ and $\mathcal{M}$ in order to construct τ.

However, there are only finitely many functions $s\colon n+1 \to \wp(n+1)$, and n is bounded by $|\varphi|$. Hence φ induces a finite set of substitution instances, one for each s function (for each $p \in \mathsf{At}(\varphi)$), in which at least one formula is falsifiable if φ is not schematically valid. This observation yields a decision procedure for the substitution core of PAL-RC. For a given φ, construct the finite set of substitution instances as described. Check the validity of each formula in the set by the standard decision procedure for PAL-RC. If φ is schematically valid, then all of its substitution instances in the set will be valid. If φ is not schematically valid, then one of the substitution instances will be falsifiable by Fact 2.

Theorem 1 (Decidability for PAL-RC). The substitution core of multi-agent PAL-RC is decidable.

Suppose that we have obtained from the PAL-RC procedure a falsifiable substitution instance $\tau(\varphi)$. Then by the effective finite model property for PAL-RC [4], we can find a finite model $\mathcal{M}$ for which $\mathcal{M}, w \nvDash \tau(\varphi)$. Since the C operator appears in the definition of $\tau(p)$, we have $\tau(\varphi) \in \mathcal{L}_{\mathsf{PAL\text{-}RC}}$. If $\varphi \in \mathcal{L}_{\mathsf{PAL}}$, we may now obtain a substitution τ' with $\tau'(\varphi) \in \mathcal{L}_{\mathsf{PAL}}$ and a model $\mathcal{M}'$ for which $\mathcal{M}', w \nvDash \tau'(\varphi)$. If there is a K_j modality that does not occur in φ, we may modify τ to τ' by replacing all occurrences of C in $\tau(\varphi)$ by K_j; then modify $\mathcal{M}$ to $\mathcal{M}'$ by setting the $\sim_j$ relation equal to the transitive closure of the union of all $\sim_i$ relations. It is straightforward to verify that $\mathcal{M}', w \nvDash \tau'(\varphi)$ given $\mathcal{M}, w \nvDash \tau(\varphi)$.

If all K_j modalities occur in φ, then we use the fact that for any finite model $\mathcal{M}$, we can define the formula $C\alpha$ in $\mathcal{M}$ by $E^{|\mathcal{M}|}\alpha$, where

$$E^1\alpha := \bigwedge_{i \in \mathsf{Agt}} K_i\alpha \text{ and } E^{n+1}\alpha := EE^n\alpha.$$

Hence we modify τ to τ' by replacing all occurrences of $C\alpha$ in $\tau(\varphi)$ by $E^{|\mathcal{M}|}\alpha$. It is straightforward to verify that $\mathcal{M}, w \nvDash \tau'(\varphi)$ given $\mathcal{M}, w \nvDash \tau(\varphi)$.

Theorem 2 (Decidability for PAL). The substitution core of multi-agent PAL is decidable.

3 Conclusion

In this paper, we have answered positively the open question [3,2,4] of whether the substitution cores of multi-agent PAL and PAL-RC are decidable. In a continuation of this work [14], we will show that our approach to proving decidability applies not only when interpreting the languages of PAL and PAL-RC in models with equivalence relations, but also when allowing models with arbitrary relations. We will also present axiomatizations of the substitution cores of PAL and PAL-RC in a system of Uniform Public Announcement Logic (UPAL).

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