Fractal Completeness Techniques in Topological Modal Logic: Koch Curve, Limit Tree, and the Real Line

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July 16, 2011

Abstract

This paper explores the connection between *fractal geometry* and *topological modal logic*. In the early 1940's, Tarski showed that the modal logic $S4$ can be interpreted in topological spaces. Since then, many interesting completeness results in the topological semantics have come to light, and renewed interest in this semantics can be seen in such recent papers as [1], [2], [3], [4], and [7]. In this paper we introduce the use of fractal techniques for proving completeness of $S4$ and non-trivial extensions of $S4$ for a variety of topological spaces. The main results of the paper are completeness of $S4$ for the binary tree with limits (or *Wilson* tree), and completeness of $S4$ for the *Koch Curve*, a well known fractal curve. An important corollary is a new and very much simplified proof of completeness of $S4$ for the real line, $\mathbb{R}$ (originally proved by Tarski and McKinsey in [9]). These results are best seen as examples of the power of the fractal techniques introduced. The main technique is developed to relate formally the somewhat peculiar non-Hausdorff tree topologies with more familiar Euclidean and other metric topologies. As we argue in the paper, the techniques developed here can be usefully applied to a wide range of completeness problems in the topological semantics.

This paper explores the connection between *fractal geometry* and *topological modal logic*. The main result of the paper is a proof of completeness of the modal logic $S4$ with respect to the *Koch Curve*, a well known fractal curve. An important corollary is a new proof of completeness of $S4$ for the real line, $\mathbb{R}$. The latter result was originally obtained by Tarski and McKinsey in [9], and much simplified and refined by Mints *et al.* and van Benthem *et al.*[1]. Our new proof uses fractal techniques, that, as we will argue in the last section, are the paper's main contribution to the topological semantics for modal logic. Completeness for both the Koch Curve and $\mathbb{R}$ are best seen as examples of the power of the fractal techniques introduced. The results of Section 4 and the techniques above are not tailor-made for solving completeness of S4 for the real line or for the slightly wider problem of completeness of S4 with respect to interesting classes of metric topological models. Rather, they should be seen as tools for obtaining completeness results for a larger variety of languages and with respect to the full range of Euclidean and other metric topologies. The main technique is developed to relate formally the somewhat peculiar non-Hausdorff tree topologies with more familiar Euclidean

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[1]Examples, variations, and refinements of this proof can be found in [1], [4], and [7].

and other metric topologies. As we will see, completeness is transferred from an appropriate tree to a metric space by means of a known fractal curve.

In what follows we use the infinite binary tree with limits, the *Wilson tree*. We prove that $S4$ is complete for the Wilson tree, and leverage this result toward a proof of completeness of $S4$ for the Koch curve and the real interval $[0,1]$. The Wilson tree is a combination of two formal relatives: the tree of all finite strings of 0's and 1's and the set of all countably infinite strings of 0's and 1's. The Wilson tree is the union of the two. Although this tree has not, to our knowledge, been used in modal logic, it has come to our attention since the time of writing this paper that the tree was introduced in category theory by Peter Freyd in the late 1980s. Freyd named the tree 'Wilson,' after Bill Wilson.[2] We follow this naming convention.

The paper is organized in five sections. Section 1 introduces the basic modal language, the standard relational modal semantics, and recalls some basic completeness results. Section 2 demonstrates the use of trees as models for the modal language, and shows that $S4$ is complete with respect to the class of models over the infinite binary tree frame. Section 3 explores the topological semantics for the modal language, introduces the Wilson tree, and shows that the completeness of $S4$ extends to this tree. Section 4 is the part of the paper where we prove our main results. It introduces the Koch Curve, and simultaneously shows completeness of $S4$ for the Koch Curve and for the real interval $[0,1]$. In Section 5, we conclude by announcing some further applications of fractal techniques in the topological semantics. A reader familiar with modal logic can skim through much of Sections 1 and 2. Furthermore, a reader familiar with the topological semantics for modal logic can leaf through all but the proof of completeness of $S4$ for the class of models over the Wilson tree in Section 3. If the reading seems somewhat terse in places, sufficient background information can be obtained by reading the excellent and very current summary of the state of topological modal logic in [2]. Many of the results and techniques that are briefly explained in our paper are properly expounded on there.

1 Modal Language, Trees, and Topology

1.1 Modal Language, Models, and Truth

Let the modal language L consist of a countable set, $\mathbb{P} = \{P_i \mid \text{ for all } i \in \mathbb{N}\}$, of atomic variables and be closed under binary connectives $\rightarrow, \vee, \wedge$ and unary operators $\neg, \Box, \Diamond$.

A *frame* is an ordered pair, $\mathcal{F} = (U, R)$, where U is a set of points called the *universe*, and R is a binary relation on U. We say $\mathcal{F}$ is *transitive* (reflexive) if R is transitive (reflexive). We interpret L in a *model* $\mathcal{M} = (\mathcal{F}, V)$, where $\mathcal{F}$ is a frame, and $V : \mathbb{P} \rightarrow \wp(U)$ is a *valuation* function.

Formulas are interpreted on points $x \in U$ and we write $\mathcal{M}, x \models \phi$ to mean that in the model $\mathcal{M}$ at the point x, ϕ holds. More specifically for a model $\mathcal{M} = ((U, R), V)$ and a point $x \in U$, the ternary relation $\mathcal{M}, x \models \phi$ is interpreted inductively as follows:

[2]Bill Wilson was the founder of Alcoholics Anonymous. It was Peter Freyd himself during a presentation of our work at The University of Pennsylvania who recognized the similarity between the tree we were studying, and a tree that he used in a result some 20 years prior. Curiously, Freyd had also used the tree as an intermediary between the standard infinite binary tree and the real interval [0, 1], albeit with a somewhat different purpose in mind.

For $P \in \mathbb{P}$,

$\mathcal{M}, x \models P \Leftrightarrow x \in V(P)$

$\mathcal{M}, x \models \neg\phi \Leftrightarrow \mathcal{M}, x \not\models \phi$

$\mathcal{M}, x \models (\phi \to \psi) \Leftrightarrow \mathcal{M}, x \not\models \phi$ or $\mathcal{M}, x \models \psi$

$\mathcal{M}, x \models \Box\phi \Leftrightarrow \mathcal{M}, y \models \phi$ for all y such that Rxy

$\mathcal{M}, x \models \Diamond\phi \Leftrightarrow \mathcal{M}, y \models \phi$ for some y such that Rxy.

The interpretation for $\vee, \wedge$ and $\leftrightarrow$ can be obtained from $\to$ and $\neg$ via the standard definitions. We could have defined $\Diamond P$ as $\neg\Box\neg P$ but the definition was added for the completeness of presentation.

Definition 1.1 (Logic S4) *The modal logic $S4$ in the language L consists of some complete axiomatization of classical propositional logic PL, some complete axiomatization of the minimal normal modal logic K, say the axiom:*

$C : (\Box P \wedge \Box Q) \to \Box(P \wedge Q)$,

and the rules:

RN: $\vdash \phi \Rightarrow \vdash \Box\phi$, *and*

RM: $\vdash \phi \to \psi \Rightarrow \vdash \Box\phi \to \Box\psi$;[3]

and, finally, the special S4 axioms:

$4 : \Box P \to \Box\Box P$

$T : \Box P \to P$

We define standard validity relations. Let $\mathcal{F} = (U, R)$ be a frame, and let $\mathcal{M} = (\mathcal{F}, V)$ be a model over $\mathcal{F}$. For any formula $\phi \in L$, we say ϕ is *true* in $\mathcal{M}$ if $\mathcal{M}, x \models \phi$ for all $x \in U$. We say ϕ is *valid in* $\mathcal{F}$ if ϕ is true in every model over $\mathcal{F}$. If $\mathcal{C}$ is a class of frames, we say ϕ is *valid in* $\mathcal{C}$ if ϕ is valid in every frame in $\mathcal{C}$. Finally, the logic $S4$ is *complete* for $\mathcal{C}$ if every formula valid in $\mathcal{C}$ is a theorem of $S4$ (*i.e.*, can be derived from the axioms together with the rules of inference). With slight abuse of notation, we will sometimes say that $S4$ is complete for for a single frame $\mathcal{F}$, where we mean $S4$ is complete for $\{\mathcal{F}\}$.

1.2 Kripke's classic completeness results

Definition 1.2 (Rooted Frames and Models) *A* rooted *(or* pointed*) frame is a triple, $\mathcal{F} = (U, R, x)$, where (U, R) is a frame, $x \in U$, and for all $y \in U$, $(x, y) \in R$.*

That is, the point x is R-related to every other point in U (or x "sees" all $y \in U$, for short).

[3] This somewhat unusual axiomatization of K and hence of $S4$ makes the topological connection introduced later on in the paper more explicit. C interpreted topologically states that the intersection of opens is open, RN states that the universe is open, RM states that if P is a subset of Q, then the interior of P is a subset of the interior of Q. Furthermore, T states that the interior of P is a subset of P, and, finally, 4, together with T, states that the interior of the interior of P is just the interior of P. This should strongly remind the reader of Kuratowski's axiomatization of the interior operator.

Theorem 1.3 *[Kripke]*

The modal logic S4 is sound and complete for (i) the class of all transitive, reflexive frames; (ii) the class of all finite transitive, reflexive frames; (iii) the class of all rooted, finite, transitive, reflexive frames.

We will not reproduce this classic result here. Most standard introductory presentations of modal logic contain proofs of (*i*), (*ii*), and (*iii*). For Kripke's original proof we refer the reader to [8]; for a more contemporary variant, see [5].

In the next section we recall that the infinite binary tree, T_2, with a transitive, reflexive relation, R_2, can be used to build models for the modal language. Indeed, the logic $S4$ is complete for the class of models over the frame T_2: a modal formula ϕ is a theorem of $S4$ if and only if it is valid in every model over T_2. Below, we show how to view T_2 (and, for that matter, *any* transitive, reflexive frame) as a topological space. We then introduce an uncountable topological extension of T_2 that we call $\mathbb{T}_2^+$. This new structure extends T_2 by adding to it uncountably many "limit nodes," corresponding to each (infinite) branch of T_2. Our main contribution to the theory of tree topologies is the proof that $S4$ is complete for $\mathbb{T}_2^+$. As we mentioned above, the significance of $\mathbb{T}_2^+$ for us lies in large part in its use in extending topological completeness results to various metric and fractal spaces. We start with a brief discussion of T_2 viewed as a relational frame.[4]

2 A modal view of T_2

2.1 The infinite binary tree, T_2

Let $\Sigma = \{0, 1\}$, and let Σ^* be the set of all finite strings over Σ including $\langle\cdot\rangle$, the empty string. Let Σ^o be the set of all countably infinite strings over Σ, and let $\Sigma^+ = \Sigma^* \cup \Sigma^o$. For $x, y \in \Sigma^*$, let $x * y$ denote the concatenation of x and y. We will also write xy for $x * y$. Concatenation is further defined for $x \in \Sigma^*$ and $y \in \Sigma^o$, but not for $x, y \in \Sigma^o$.

Σ^* is closed under concatenation, that is, if $x, y \in \Sigma^*$ then $x * y \in \Sigma^*$. Similarly, Σ^+ is closed under "right-concatenation" in the following sense: for $x \in \Sigma^*$, $y \in \Sigma^+$, $x * y \in \Sigma^+$.

We let $s_i : \Sigma^* \to \Sigma^*$ for $i \in \{0, 1\}$ be the function defined by $s_i(x) = x * i$. Thus for example $s_0(1) = 10$, and $s_1(110) = 1101$. We call $s_0(x)$ the "left successor" of x and $s_1(x)$ the "right successor" of x.

Define the binary relation R_2 on Σ^* as the transitive reflexive closure of $s_0 \cup s_1$ (where s_i is viewed here as a relation, rather than a function).

Then,

Definition 2.1 (T_2, a modal frame) $T_2 = (\Sigma^*, R_2, \langle\cdot\rangle)$

We call T_2 the *infinite binary branching tree* or *full binary tree*. We call the empty string, $\langle\cdot\rangle$, the *root*, and for any $x \in T_2$, $s_0(x)$ and $s_1(x)$ are called the *immediate successors* of x. For simplicity of notation, we will often leave out the root, $\langle\cdot\rangle$, denoting T_2 by (Σ^*, R_2).

[4] The formal details of the next section follow the presentation in [1]. The details can be skipped by a reader familiar with the notion of *tree unravelling*.

Fact 2.2 *Every node x is accessible from the root in finitely many steps along R_2 and hence in one step by transitivity. Every $x \in T_2$ has exactly two immediate successors and countably many successors altogether.*

A valuation function $V : \mathbb{P} \to \wp(\Sigma^*)$ defines a model $\mathcal{T}_2$ over T_2. Since T_2 is transitive and reflexive any such model validates the $S4$ axioms – *i.e.*, $S4$ is sound for T_2.

Claim 2.3 *For any finite, transitive, reflexive, rooted, model $\mathcal{M}$, with root x, there is a valuation V over T_2 such that,*

$$\mathcal{M}, x \models \phi \Leftrightarrow (T_2, V), \langle\cdot\rangle \models \phi$$

for every $\phi \in L$.

(The proof of the claim is postponed until the next section.)

It follows from Claim 2.3 and Theorem 1.3 that every nontheorem of $S4$ can be shown false on some model based on the frame T_2. Indeed, if ϕ is not a theorem of $S4$, then by Theorem 1.3, there is some finite rooted frame $\mathcal{F} = (U, R, x)$ and valuation V such that $(\mathcal{F}, V), x \models \neg\phi$. But then by Claim 2.3, there is a valuation V' over T_2 such that $(T_2, V'), \langle\cdot\rangle \models \neg\phi$. Thus any nontheorem fails on T_2, and $S4$ is complete for the class of models over T_2.

2.2 Building a $p-$morphism from T_2 onto a finite frame $\mathcal{F}$

We prove Claim 2.3 by constructing a $p-$morphism $f : T_2 \to \mathcal{F}$, where $\mathcal{F} = (U, R, x)$ is a finite, rooted, transitive and reflexive frame. We briefly recall the notion of $p-$morphism.

Definition 2.4 (p-morphism) *Let $\mathcal{F} = (U, R, x_r)$ and $\mathcal{F}' = (U', R', x'_r)$ be rooted frames. A p-morphism from $\mathcal{F}$ to $\mathcal{F}'$ is a function $f : U \to U'$ satisfying: For any $x, y \in U$ and $y' \in U'$,*

(0) $f(x_r) = x'_r$;

(i) If Rxy, then $f(x)R'f(y)$;

(ii) If $R'f(x)y'$, then there is a $z \in U, Rxz$ and $f(z) = y'$.

We say that f is a surjective p-morphism *if, in addition, $f(U) = U'$.*

Fact 2.5 *If there is a surjective $p-$morphism f from $\mathcal{F}$ to $\mathcal{F}'$, then for any valuation function $V : \mathbb{P} \to \wp(U')$, any point $x \in U$, and any modal formula ϕ, we have:*[5]

$$(\mathcal{F}, [f^{-1}] \circ V), x \models \phi \Leftrightarrow (\mathcal{F}', V), f(x) \models \phi$$

Thus, to prove Claim 2.3 it suffices to show that for any finite, transitive, reflexive, rooted frame $\mathcal{F} = (U, R, x)$, there is a surjective $p-$morphism f from T_2 to $\mathcal{F}$.

[5] The function $[f^{-1}] : \wp(U') \to \wp(U)$ raises the type: for $A \subseteq U'$, $[f^{-1}](A) = \{y \mid f(y) \in A\}$. Note that although f^{-1} is likely not a function, $[f^{-1}]$ is always a function, but of a higher type. Thus, the function $[f^{-1}] \circ V : \mathbb{P} \to \wp(U)$, i.e., it is a valuation function.

Let the cardinality of U in $\mathcal{F}$ be n. Notice that no point in U has more than n distinct successors and x, the root, actually has n successors. We now construct the function f.

For $1 \leq i \leq n (= |U|)$, we define the set of functions $s_i : U \to U$ $(1 \leq i \leq n)$. For each $y \in U$, the function s_i chooses the ith distinct $R-$successor of y, if such a successor exists. Otherwise $s_i(y) = y$. More formally,

Definition 2.6 (Successor functions s_i) *For all y, $s_1(y) = y$ (s_1 is the identity function). Fix $i \in \mathbb{N}$, and suppose that $s_1(y)$, $s_2(y)$, ..., $s_{i-1}(y)$ are already defined, and that $Rys_k(y)$ for all $k < i$. Then we let $s_i(y)$ be some $z \in U$ such that Ryz and $s_k(y) \neq z$ for all $k < i$, if there is some such z. Else, $s_i(y) = y$.*

Example 2.7 (A set of successor functions) *Let $y \in U$ have 3 distinct successors including y itself: y, w and z and no others. Then if $|U| = 5$, we let $s_1(y) = y$, $s_2(y) = w$, $s_3(y) = z$, but $s_4(y) = s_5(y) = y$ as we have run out of distinct successors.*

Definition 2.8 *[Unraveling $p-$morphism]*

We define a linear ordering on the nodes in T_2. This can be done in many ways, but for specificity, we let, e.g., $\langle\cdot\rangle < 0 < 1 < 00 < 01 < 10 < 11 < 000 < ...$

[base step.] First let $f(\langle\cdot\rangle) = x$.

[recursive step.] Until f is defined for all nodes in T_2, find the least[6] node t such that $f(t)$ is defined, but neither $f(t * 0)$ nor $f(t * 1)$ is defined. Assume that $f(t) = y$. Then let,

$$f(t * 1) = s_1(y), \;\; f(t * 01) = s_2(y), \;\; f(t * 001) = s_3(y), \;\; ... \;\; f(t * 0^{n-1} * 1) = s_n(y)$$

where 0^{n-1} is a sequence of $n - 1$ zeros. Finally, let,

$$f(t * 0) = f(t * 00) = f(t * 000) = ... = f(t * 0^n) = s_1(y) = y.$$

Lemma 2.9 *[Unravelling Lemma] Let f be the function defined in Definition 2.8. Then f is a $p-$morphism.*

Proof. (i) It suffices to show that if R_2st and t is the immediate successor of s, then $Rf(s)f(t)$. This can be seen by inspecting the recursive step of Definition 2.8. If $f(s) = y$, then $f(t)$ is $s_i(y)$, for some $i \in \{1, ..., n\}$, but, by definition of s_i, we know $Rys_i(y)$ for each such i. (ii) We need to show that if $Rf(t)z$, then there exists $s \in T_2$ such that R_2ts, and $f(s) = z$. We let $f(t) = y$ and recall that $s_1(y), s_2(y), ..., s_n(y)$ exhaust the distinct $R-$successors of y in $\mathcal{F}$. Then for some $i \in \{1, ..., n\}$, $s_i(y) = z$. If t was ever the least node satisfying the antecedent condition of Definition 2.8, then some successor of t was labeled by $s_i(y)$ – *i.e.*, by z. Otherwise, t is a successor of some other node t', which did at some stage satisfy the antecedent condition of Definition 2.8 and $t = t' * 0^k$ for some $k \leq n$. But then, at that stage, for some successor t'' of t, $f(t'') = y$ and $t'' * u$ was undefined for any nonempty finite sequence u. Thus at some future stage a successor of t'' was labeled with $s_i(y)$ (i.e. z). But a successor of t'' is a successor of t by transitivity of R_2, as desired.

Putting Fact 2.5 and Lemma 2.9 together, we obtain the desired completeness result:

[6] On the ordering just given.

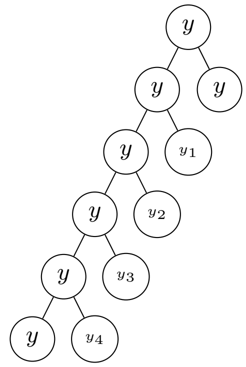

Figure 1: The recursive step of the definition of the $p-$morphism f. Here $|U| = 5$, $f(t) = y$, $s_1(y) = y$, $s_2(y) = y_1$, $s_3(y) = y_2$, $s_4(y) = y_3$, and $s_5(y) = y_4$. Following the definition, $f(t*01) = y_1$, $f(t*001) = y_2$, $f(t*0001) = y_3$, $f(t*00001) = y_4$, and all other points visible in the diagram are labeled y. No successor of t except for the eleven nodes (really ten and t) explicitly shown in the diagram is labeled at this stage.

Fact 2.10 *The modal logic $S4$ is complete for the class of models over the frame $T_2 = (\Sigma^*, R_2, \langle\cdot\rangle)$.*

In the next section we look at modal language L and the frame $T_2 = (\Sigma^*, R_2)$ from a topological perspective.

3 Topological Semantics for Modal Logic

We now turn to topology and the topological interpretation of the modal language L. Long before Kripke-semantics for the modal language was established as the yardstick, A. Tarski and J.C.C. McKinsey noted an irresistible connection between Lewis and Langford's axioms for the modal logic $S4$, and Kuratowski's axioms for the topological interior operator. The topological interpretation of modal logic exploits this connection .[7]

Tarski's idea was to view $\Box A$ as the interior of the set A and $\Diamond A$ as the closure of A and try to understand what kind of logical structure such an interpretation supported. Tarski was able to prove – in some sense quite unsurprisingly – that under this interpretation the logic of the interior and closure operators turns out to be nothing less than $S4$. The argument for the general case is straightforward, as we'll see below. The arguments for specific topological spaces turn out to be rather more involved. It is part of our goal here to try to understand where such complexity comes from. Let us introduce some basic background notions.

[7]Equivalently, one can go by the connection between the $\Diamond$-version of the $S4$ axioms and the behavior of the *closure* operator C, via the definition $I(A) = -C(-A)$. (The interior of a set is just the complement of the closure of the complement).

3.1 Topological semantics for modal language L and the connection with Kripke frames

A topology is a set of points with some spatial structure (one can think of it as a set of points glued together in a certain way). Specifically, a topology is a pair, $\mathcal{X} = (X, \mathcal{J})$, where X is a set and $\mathcal{J} \subseteq \wp(X)$ satisfies,

1. $X, \emptyset \in \mathcal{J}$,

2. If $A, B \in \mathcal{J}$, then $A \cap B \in \mathcal{J}$,

3. If $A_i \in \mathcal{J}$ for all $i \in I$, then $\bigcup_{i \in I} A_i \in \mathcal{J}$.

If in addition a topology satisfies,

4. If $A_i \in \mathcal{J}$ for all $i \in I$, then $\bigcap_{i \in I} A_i \in \mathcal{J}$

then the topology is called *Alexandroff*. As we'll see, most interesting topologies are *not* Alexandroff. More (structure) is not always better as a cursory comparison between Italian and American pizza quickly reveals.

Although a topological space is strictly speaking a pair, $(X, \mathcal{J})$, we will for simplicity of notation (and where the meaning is clear) often denote both the topological space itself and the underlying set of points by X. The sets in $\mathcal{J}$ are called *open* sets. We say a set is *closed* if its complement is open. The union of open subsets of a set, A, is called the *interior* of A:

$$Int(A) = \bigcup \{O \text{ open} \mid O \subseteq A\}$$

The *closure* of a set is the complement of the interior of the complement:

$$Cl(A) = -Int - (A)$$

(Equivalently, a point x is in the closure of A if every open set containing x contains some element in A.)

We wish to interpret our language L in topological models. A topological model $\mathcal{T}$ is a pair $(\mathcal{X}, V)$ where $\mathcal{X} = (X, \mathcal{J})$ is a topology and $V : \mathbb{P} \to \wp(X)$ is a valuation function. We define a ternary relation $\mathcal{T}, x \models \phi$ that as before holds between a point in a model and a formula. The cases for the atomic and Boolean formulae are the same. The only real difference is in the modal cases of $\Box$ and $\Diamond$. We want $\Box\phi$ to be true at a given point x if x is in the interior of the set defined by the formula ϕ. Then also $\Diamond\phi$ should hold at x if x is in the closure of the set defined by ϕ . We encode these observations in the following truth definitions:

$\mathcal{T}, x \models \Box\phi \Leftrightarrow \exists O \in \mathcal{J} \text{ such that } x \in O \text{ and } \forall y \in O, \mathcal{T}, y \models \phi.$

$\mathcal{T}, x \models \Diamond\phi \Leftrightarrow \forall O \in \mathcal{J}, x \in O \text{ implies } \exists y \in O \text{ such that } \mathcal{T}, y \models \phi.$

Let $\mathcal{X} = (X, \mathcal{J})$ be Alexandroff and let $x \in X$. Consider the set $O_x = \bigcap \{O \in \mathcal{J} \mid x \in O\}$, i.e., the intersection of all open sets containing x. Note that since our topological space is Alexandroff, this is a non-empty open set. We define the binary relation R on X:

$$Rxy \Leftrightarrow y \in O_x.$$

Then

Claim 3.1 *$\mathcal{F}_{\mathcal{X}} = (X, R)$ is a reflexive, transitive frame.*

Proof. For reflexivity, note that $x \in O_x$. For transitivity, suppose Rxy and Ryz. Then $y \in O_x$ and $z \in O_y$. From the first inclusion it follows that $O_y \subseteq O_x$. So we have $z \in O_y \subseteq O_x$, and hence Rxz.

Moving in the reverse direction, we can generate a topology from a reflexive, transitive frame. Let $\mathcal{F} = (X, R)$ be a reflexive, transitive frame. Let $\mathcal{J}_{\mathcal{F}}$ be the collection of subsets of X that are upward-closed under R (where a set $O \subseteq X$ is upward-closed under R if $x \in O$ and Rxy implies $y \in O$). Then,

Claim 3.2 *$\mathcal{X}_{\mathcal{F}} = (X, \mathcal{J}_{\mathcal{F}})$ is Alexandroff.*

Proof. The reader can verify that $\mathcal{X}_{\mathcal{F}}$ is a topological space. To see that it is Alexandroff, suppose that $\{O_i \mid i \in I\}$ is a collection of open sets in the topology and let $x \in \bigcap_{i \in I} O_i$ and Rxy. Then since each O_i is upward-closed under R, $y \in O_i$ for each $i \in I$. But then $y \in \bigcap_{i \in I} O_i$, and $\bigcap_{i \in I} O_i$ is upward-closed under R, as desired.

The reader is invited to verify that the operations of generating a transitive, reflexive frame from an Alexandroff topology, and of generating an Alexandroff topology from a transitive, reflexive frame just described are inverses of one another: if one starts with an Alexandroff topology, then generates a transitive reflexive frame, and then, from this frame, generates an Alexandroff topology in the manner described, one ends up with the original topological space (and similarly, when one starts from a transitive, reflexive frame). When a frame and topological space are generated in this way by one another, we will sometimes say they "correspond." The next proposition states that corresponding frames and topological spaces satisfy the same modal formulas:

Proposition 3.3 *Let $\mathcal{X} = (X, \mathcal{J})$ be an Alexandroff topology and let $\mathcal{F} = (X, R)$ be a transitive, reflexive frame. If $\mathcal{X}$ and $\mathcal{F}$ correspond, then for any formula $\phi \in L$, any $x \in X$, and any valuation $V : \mathbb{P} \to \wp(X)$*

$$(\mathcal{F}, V), x \models \phi \Leftrightarrow (\mathcal{X}, V), x \models \phi$$

Proof. The proof is by induction on the complexity of ϕ. We show only the modal clause, $\phi :\equiv \Box\psi$. We have,

$$\begin{aligned}
(\mathcal{F}, V), x \models \Box\psi &\Leftrightarrow (F, V), y \models \psi \text{ for all } y \text{ such that } Rxy \\
&\Leftrightarrow (\mathcal{F}, V), y \models \psi \text{ for all } y \in O_x \\
&\Leftrightarrow (\mathcal{X}, V), y \models \psi \text{ for all } y \in O_x \qquad \text{(by Induction Hypothesis)} \\
&\Leftrightarrow (\mathcal{X}, V), x \models \Box\psi
\end{aligned}$$

What these observations tell us is that Alexandroff topologies are nothing more than reflexive, transitive frames. This is both useful and limiting. On the positive side, it allows us to transfer a variety of important results directly to the topological semantics. On the negative side, most interesting topologies are non-Alexandroff (e.g. metric spaces). Much of our work in what follows will be constructing "nice" maps between metric spaces and non-Alexandroff topologies.

3.2 Interior maps and truth preservation in topological semantics

The work in the sections below requires us to recall some additional topological notions. In the topological semantics, the notion of an *interior map* plays the role of $p-$morphism in the Kripke (or frame) semantics. In fact, when the topologies in question are Alexandroff, the notions of $p-$morphism and interior map correspond exactly.

Let $\mathcal{X} = (X, \mathcal{J}_X)$ and $\mathcal{Y} = (Y, \mathcal{J}_Y)$ be topological spaces.

Definition 3.4 (Open Map) *A map $g : \mathcal{X} \to \mathcal{Y}$ is* open *if for every set $O \in \mathcal{J}_x$, $g(O) \in \mathcal{J}_y$.*

Definition 3.5 (Continuous Map) *A map $g : \mathcal{X} \to \mathcal{Y}$ is* continuous *if for any set $U \in \mathcal{J}_y$, $g^{-1}(U) \in \mathcal{J}_x$.*

Definition 3.6 (Interior Map) *A map $g : \mathcal{X} \to \mathcal{Y}$ is* interior *if it is both open and continuous.*

Definition 3.7 (Full-Interior Map) *We will say a map $g : \mathcal{X} \to \mathcal{Y}$ is* full-interior *if it is interior and surjective.*

Fact 3.8 (Full-Interior Maps Preserve Modal Formulas) *Let $g : \mathcal{X} \to \mathcal{Y}$ be a full-interior map, and ϕ any formula of the basic modal language L. Let $V' : \mathbb{P} \to \wp(Y)$ be a valuation function and let $V = ([g^{-1}] \circ V')$.*[8] *Then, for any $x \in X$,*

$$(\mathcal{X}, V), x \models \phi \Leftrightarrow (\mathcal{Y}, V'), g(x) \models \phi$$

Proof. The proof is by induction on the complexity of ϕ. The base case and the Boolean cases are straightforward. For the modal case:

$$(\mathcal{X}, V), x \models \Box\psi \Leftrightarrow (\mathcal{Y}, V'), g(x) \models \Box\psi$$

we use the preservation of open sets along g to show the left-to-right direction, and we use the continuity of g to show the right-to-left direction. The details of the proof can be found in, *e.g.*, [1].

Now suppose that $\mathcal{X} = (X, \mathcal{J}_X)$ and $\mathcal{Y} = (Y, \mathcal{J}_Y)$ are *Alexandroff* topologies, and let $\mathcal{F}_{\mathcal{X}}$ and $\mathcal{F}_{\mathcal{Y}}$ be the corresponding frames. Moreover, let $g : \mathcal{X} \to \mathcal{Y}$ be a full-interior map. Then,

Fact 3.9 *The function g reinterpreted as $g : \mathcal{F}_{\mathcal{X}} \to \mathcal{F}_{\mathcal{Y}}$ is a $p-$morphism.*

Proof. See *e.g.*, [1].

Just as $p-$morphisms play an important role in transferring completeness results in the relational semantics, interior maps play a similar role in transferring completeness results in the topological semantics. In the remainder of this section, we recall some of the better known topological completeness results for $S4$. We then use a particular sequence of interior maps to prove completeness for the Koch fractal and the real interval $[0, 1]$.

[8] Thus V is a valuation function on X, defined as the composition of g^{-1} with V'.

3.3 Topological completeness results for $S4$

Theorem 3.10 *The logic S4 is sound and complete with respect to*

(i) the class of all topologies (McKinsey & Tarski);

(ii) the class of all finite topologies (Kripke);

(iii) any dense-in-itself metric space (including, e.g., *$\mathbb{R}^n$, for any $n \in \mathbb{N}$) (McKinsey &Tarski);*

(iv) the infinite binary tree, $\mathbb{T}_2$ (see below) (van Benthem, Gabbay).

In this paper we will show,

(v) a direct construction for the singleton class K, the Koch Curve. *The Minkowski–Bouligand dimension of K is 1.26. (This paper or McKinsey &Tarski).*[9]

(vi) the Willson Tree or Infinite Binary Tree with Limits, $\mathbb{T}_2^+$, equipped with the topology generated by finite initial segments [see Definition 3.11]. (This paper)

Proof. (*ii*) follows from completeness for finite frames; (*iii*) is proved in [9]; (*i*) follows from either (*ii*) or (*iii*); (*iv*) follows from Lemma 2.9, originally due to van Benthem and Gabbay.[10] For (*v*) and (*vi*), see the later sections of this paper.

Part of our goal in this paper is to revisit (*iii*) – in particular, the special case of $\mathbb{R}$ – as well as to extend the topo-completeness results to the Koch Snowflake fractal. We will also mention some other fractals that are useful in topo-modal constructions and for which completeness results can be had. We have in mind, in particular, a direct proof of completeness of $S4$ for $\mathbb{R}^2$ and $\mathbb{R}^3$ via the Sierpinski Carpet and Menger Sponge, respectively.

3.4 A topological perspective on T_2, and the extension to the limit tree topology $\mathbb{T}_2^+$

The infinite binary tree, T_2, is a rare object in mathematics that exhibits interesting structural features from a great range of different perspectives. As we saw above, it has enough structural symmetry and flexibility to carry the weight of the completeness theorem of $S4$ in the relational semantics. T_2 recurs when we start thinking of space fractally. We look next at an extension of T_2 called the Wilson tree, or infinite binary tree with limits, that allows us to prove two completeness results in the topological semantics.

3.4.1 Wilson Tree (Infinite Binary tree with Limits), $\mathbb{T}_2^+$

Definition 3.11 *Take alphabet $\Sigma = \{0, 1\}$ and construct the set $\Sigma^*(\Sigma^+)$ of all finite (countable) strings over Σ. For any $s \in \Sigma^*$, let $B_s = \{s * t \,|\, t \in \Sigma^+\}$, i.e. the set of all (possibly infinite) strings*

[9] Since the standard topological dimension of K is 1, there is a homeomorphism h between K and $[0, 1]$. Thus, we know that $S4$ is complete for K as we can transfer counterexamples via h. However, this is the first direct completeness construction on a fractal curve of non-integer Minkowski–Bouligand dimension, except for Cantor Set.

[10] Both Johan van Benthem and Dov Gabby introduced a variant of the unravelling technique, but the historical precedence, as far as we know, is unclear.

with initial segment s (where s is allowed to be the empty string). Let $B = \{B_s \mid s \in \Sigma^*\}$*. Note that* B *is closed under finite intersections (For any* $s, t \in \Sigma^*$ *either* $B_s \subseteq B_t$*,* $B_t \subseteq B_s$*, or* $B_s \cap B_t = \emptyset$*), hence is a basis for some topology* $\mathcal{J}^+$ *over* Σ^+*. Finally, let* $\mathbb{T}_2^+ = (\Sigma^+, \mathcal{J}^+)$*.*

Fact 3.12 *(i)* Σ^+*, the underlying set of* $\mathbb{T}_2^+$*, is uncountable;*

(ii) $\mathbb{T}_2^+$ *is first countable;*

(iii) $\mathbb{T}_2^+$ *is non-Alexandroff.*

Separation axioms*:*

(iv) $\mathbb{T}_2^+$ *is* T_0*,*

(v) $\mathbb{T}_2^+$ *is not* T_1 *(hence non-Hausdorff and non-metrizable)*

(i) follows from an injection between the set of countably infinite strings over Σ and the real interval [0, 1]; (ii) follows from the fact that the basis, B, is countable; (iii) the intersection of basic opens $B_0, B_{00}, B_{000}, ...$ (i.e. the countable sequence 000...) is not open; (iv) For $s, t \in \Sigma^+$, $s \neq t$: if s is a descendant of t, then either B_s separates s and t (if $s \in \Sigma^*$) or there exists $t' \in \Sigma^*$ which is a descendant of t such that $B_{t'}$ separates s and t (and vice versa, if t is a descendant of s). If neither s nor t is a descendant of the other, there exists $t' \in \Sigma^*$ such that t' is an ancestor of s but not of t, and $B_{t'}$ separates s and t; (v) take, for instance, $s = 0$ and $t = 00$: there is no open set containing s that does not contain t.

In the remainder of this section, we show that $S4$ is complete for $\mathbb{T}_2^+$. To this end, recall the map $f : T_2 \to \mathcal{F} = (U, R, x)$ given in Definition 2.8. We view this function now as a map, $f : \Sigma^* \to U$, between underlying sets, and extend it to a map, $f^+ : \Sigma^+ \to U$. Moreover, we now view the frames $\mathcal{F}$ and $\mathbb{T}_2^+$ as topological spaces, and the map f^+ as a topological map. We show that f^+ is full-interior. Since $S4$ is complete for (the class of models over) finite, transitive, reflexive frames, it follows from Fact 3.8 that $S4$ is also complete for (the class of models over) $\mathbb{T}_2^+$.

We will need a few simple infinitary notions. We begin by defining an infinite branch **b** of the tree T_2.

Definition 3.13 (Countable Branch) *Let* $b =< t_0, t_1, ... >$ *be an infinite branch of* T_2*. That is:*

(i) $t_0 = \langle \cdot \rangle$*;*

(ii) For all $n = 0, 1, 2...$*, either* $t_{n+1} = t_n * 0$ *or* $t_{n+1} = t_n * 1$*.*

Lemma 3.14 *[Cycling Lemma]*

Let f *be any function from* T_2 *onto* $\mathcal{F} = (U, R, x)$*, and let* $b =< t_0, t_1, ... >$ *be an infinite branch in* T_2*. Then there exists* $N \in \mathbb{N}$ *such that for all worlds* $x \in U$*,*

$$\exists m > N \text{ such that } f(t_m) = x \text{ implies } f(t_m) = x \text{ for infinitely many } m > N.$$

Proof. The lemma follows from the fact that U is finite, so there are only finitely many labels in U for f to "choose" from. Labels that occur only finitely many times on a branch, occur for the last time at some finite node of T_2.

For a given branch b, let n_b be the least such $N \in \mathbb{N}$. Let $A_b = \{f(t_m) : m > n_b\}$. (Thus A_b is the collection of worlds in U that label infinitely many nodes of the branch, b, under f).

Note that the Lemma states that after some initial segment of b all nodes of b are sent by f to elements in A_b and each of these elements labels infinitely many nodes on the branch.

Fact 3.15 *For any $n \in \mathbb{N}$ and any $x \in A_b$, $\exists m > n$ such that $f(t_m) = x$.*

Proof. This follows from the fact that every element in A_b labels infinitely many nodes in b.

Definition 3.16 *[Branch Labeling]*

Let f be a $p-$morphism from T_2 onto the finite rooted frame $\mathcal{F} = (U, R, x)$. For every branch b in T_2, we let the finite choice function $C(b)$ return a choice of $y \in A_b$. Further, noting that every branch b has a unique countable sequence in Σ^* associated with it, we can think of the branches and elements of Σ^+ interchangeably. We define the extension, $f^+ : \Sigma^+ \to U$, of f as follows: Let t_b be the element in Σ^+ that corresponds to the branch b. We let $f^+(t_b) = C(b)$. Thus we label each countable string in Σ^+ with a node in $A_b \subseteq U$.

For the remainder of this paper we view f^+ and f interchangeably as a maps between topological spaces, frames, or simply underlying sets. From the context it should be clear which of these we intend. Also, we refer to 'finite' and 'limit' nodes of the tree $\mathbb{T}_2^+$, with the obvious interpretation.

Theorem 3.17 *$f^+ : \mathbb{T}_2^+ \to \mathcal{F}$ is full-interior.*

Proof. We need to show that f^+ is open, continuous, and surjective.

Open. Let $O \in \mathcal{J}^+$ be a basic open set. Then $O = B_s$ for some finite node s. Let $y = f^+(s)$ and let $D_y = \{z \in U \mid Ryz\}$. We show that $f^+(B_s) = D_y$. We know that every point in D_y labels some node in B_s by the fact that f is a p-morphism. Thus $D_y \subseteq f^+(B)$. For the reverse inclusion, let $z \in f^+(B_s)$. Then $z = f^+(t)$ for some $t \in B_s$. If t is finite then $f^+(t) = f(t) \in D_y$, where inclusion follows from the fact that f is a p-morphism. If t is a limit node, then $f^+(t) = f^+(t')$ for some finite node $t' \in B_s$ (by construction of f^+). Moreover, $f^+(t') = f(t') \in D_y$ (since t' is finite). Thus $f^+(B) \subseteq D_y$, as needed.

Continuous. Let U be an open set in $\mathcal{F}$. Let $s \in (f^+)^{-1}(U)$, and let $f^+(s) = y \in U$. We need to show there is an open set $O \subseteq \mathbb{T}_2^+$, such that $s \in O \subseteq (f^+)^{-1}(U)$. Now if s is finite, then we already know that $f^+(B_s) = D_y \subseteq U$ (by proof of *Open* above). So $s \in B_s \subseteq (f^+)^{-1}(U)$, where B_s is open. If s is a limit node, then there is some finite s' such that $f^+(s') = y$, and $Rs's$. But then $f^+(B_{s'}) = D_y \subseteq U$ and $s \in B_{s'} \subseteq (f^+)^{-1}(U)$, where $B_{s'}$ is open. This shows that $(f^+)^{-1}(U)$ is open, as needed.

Surjective. Surjectivity follows from the fact that $x \in Range(f^+)$ and f^+ is open (where x is the root of $\mathcal{F}$).

Theorem 3.18 *$S4$ is complete for $\mathbb{T}_2^+$.*

Proof. By Fact 3.8, Theorem 3.10, and Theorem 3.17.

In the next section, we construct a full-interior function from the real interval [0, 1] onto $\mathbb{T}_2^+$, via the Koch Curve. That construction gives us both completeness of $S4$ for the Koch Curve, and a new proof of completeness of $S4$ for the real interval $[0, 1]$.

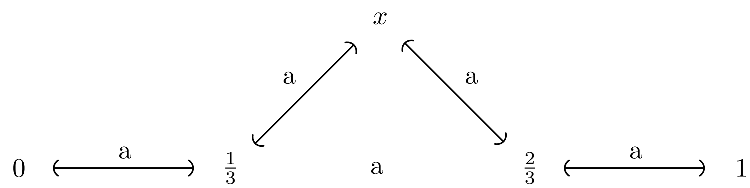

Figure 2: K_1: $a = \frac{1}{3}$

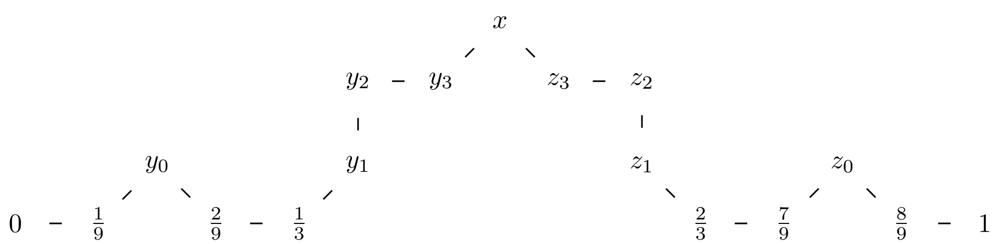

Figure 3: K_2: The length of each line segment is $\frac{1}{9}$, and the five triangles with apex's at y_0, y_2, x, z_2, z_0 are equilateral triangles

.

4 Koch Curve and Topological Completeness

Our goal is to construct a homeomorphism between the interval $[0, 1]$ and Koch Curve fractal, K, and a relatively simple full interior labelling $l : [0, 1] \to \mathbb{T}_2^+$ inspired by the construction of Koch Curve. The labeling itself provides a straightforward proof of completeness of $S4$ for the real interval. When composed with the homeomorphism we obtain completeness of $S4$ for the singleton class K, the Koch Curve.

4.1 Koch Curve

Recall the construction of the Koch curve, K. We begin with the unit interval $[0, 1]$. At the first stage, K_1, we let the middle third of the interval be "pushed up" to form two sides of an equilateral triangle with side length $\frac{1}{3}$ as pictured in Figure 2.

At the second stage we let the middle third of each line segment of K_1 be raised to form two sides of an equilateral triangle of length $\frac{1}{9}$. This gives K_2 in Figure 3.

In general, at stage n of construction, we raise the middle third of each line segment of K_{n-1} to form two sides of an equilateral triangle of side length equal to the length of the segment raised.

The Koch curve is a limit of the construction stages in the following sense. Let K_0 be the unit

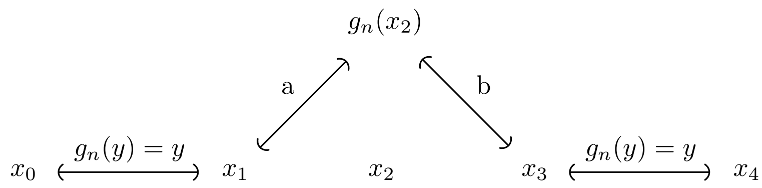

Figure 4: This figure shows how g_n acts on a single segment $[x_0, x_4]$ of K_{n-1}. g_n is the identity function everywhere except: (i) $g_n(x_2)$ is the apex of the triangle, and (ii) g_n maps the line segment (x_1, x_2) linearly onto a and maps the line segment (x_2, x_3) linearly onto b.

interval $[0, 1]$. For $n = 1, 2...$, let

$$g_n : K_{n-1} \to K_n$$

be the obvious homeomorphism from K_{n-1} to K_n.

Let

$$f_n = g_n \circ g_{n-1} ... \circ g_1$$

Thus, for each $n \in \mathbb{N}$, $f_n : [0, 1] \to K_n$ is a homeomorphism from $[0, 1]$ onto K_n. Finally, we let f be the pointwise limit of these functions:

$$f = \lim_{n \to \infty} f_n$$

and the Koch curve, K, is the range of this limit:

$$K = Range(f)$$

Claim 4.1 *$f : [0, 1] \to K$ is a homeomorphism.*

Proof. We need to show that f is bijective, continuous and open.

1. (Bijective) Note that any two distinct points $x, y \in [0, 1]$ eventually end up on different line segments under some f_n. Indeed, since $x \neq y$, we know $d(x, y) > 0$ (where d denotes the usual distance function). But the length of line segments in K_n is $(\frac{1}{3})^n$. Since $(\frac{1}{3})^n \to 0$, the length of line segments in K_n is eventually smaller than the distance $d(x, y)$, and x and y belong to different line segments. We leave it to the reader to verify that such points are not identified under f – *i.e.*, $f(x) \neq f(y)$. This shows that f is injective. Surjectivity follows from the fact that $K = Range(f)$.

2. (Continuous) We show that f is the uniform limit of continuous functions, hence continuous.[11] Note that for any $x \in [0, 1]$, $d(f_n(x), f_{n-1}(x)) = d(g_n(f_{n-1}(x)), f_{n-1}(x))$, where d denotes the distance function in the usual metric on $\mathbb{R}^2$. Moreover, by construction of g_n, g_n moves points at most a distance of $(\frac{1}{3})^n(\frac{\sqrt{3}}{2})$. So $d(f_n(x), f_{n-1}(x)) < (\frac{1}{3})^n(\frac{\sqrt{3}}{2}) \to 0$, for all $x \in [0, 1]$. Thus the f_n's converge uniformly, and the uniform limit of continuous functions is continuous.

[11] Here we view the functions f_n as functions from the space [0,1] to $\mathbb{R}^2$, with the usual metrics on each of these spaces.

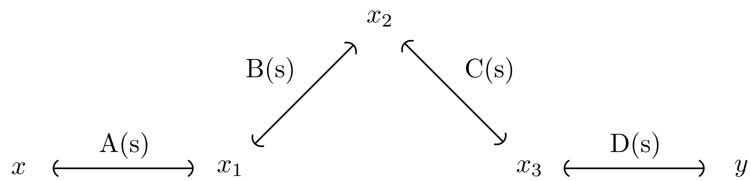

Figure 5: Segment s in K_{n-1} is $[x, y]$. Then $A(s) = [x, x_1], B(s) = (x_1, x_2)$, $C(s) = (x_2, x_3), D(s) = [x_3, y]$, and $E(s) = \{x_2\}$.

3. (Open) We first show that the image under f of a closed set is closed. Indeed, if $A \subseteq [0, 1]$ is closed, then it is also compact (since [0,1] is bounded). But the continuous image of a compact set is compact, so $f(A)$ is a compact subset of K. So $f(A)$ is closed (and bounded), as desired. Now suppose that $O \subseteq [0, 1]$ is open. Then $f(O) = f([0, 1]) - f([0, 1] - O) = K - f([0, 1] - O)$, since $f : [0, 1] \to K$ is a bijection. By the above argument, $f([0, 1] - O)$ is closed, so $f(O)$ is open.

It follows from the previous claim that $f^{-1} : K \to [0, 1]$ is a homeomorphism. We now wish to construct a function $l : [0, 1] \to \mathbb{T}_2^+$ that is full-interior. Once we have done so, l alone will prove completeness of $S4$ for the real interval [0, 1], and the composition $l \circ f^{-1} : K \to \mathbb{T}_2^+$ will prove completeness of $S4$ for the Koch curve, K. Much as we constructed f as a limit of finite approximations, f_n, we now construct the function l as a limit of stagewise labeling functions, l_n. Indeed, as the reader will presently see, the functions, l_n, correspond neatly to stages of Koch construction.

Note above that each $g_n : K_{n-1} \to K_n$ sends K_{n-1} to K_n by breaking up each line segment of K_{n-1} into four line segments of K_n. For any line segment s in K_{n-1} we refer to its "successor" segments in K_n as (in order from left to right) $A(s), B(s), C(s)$ and $D(s)$ (see Figure 5). There is an ambiguity here with respect to endpoints: is the point $\frac{1}{3}$, for example, in the segment $A([0, 1])$ or $B([0, 1])$? For reasons that will become clear below, we decide that $B(s)$ and $C(s)$ are always open on both ends, while the "right" end-point of $A(s)$ and the "left" endpoint of $D(s)$ are always closed. (The left endpoint of $A(s)$ and the right endpoint of $D(s)$ are either open or closed, depending on whether the segment s itself is open or closed at that endpoint). Thus, e.g. $\frac{1}{3} \in A([0, 1])$ and $\frac{2}{3} \in D([0, 1])$. Note that for each segment s this leaves one point still unclassified – namely, the midpoint of s which becomes in the next stage of construction, the apex of the equilateral triangle (in Figure 5, the point x_2). For simplicity, we let this one point constitute a new singleton set $E(s)$.

These definitions allow us to construct stages of labeling in a natural way. Fix $x \in [0, 1]$, $n \in \mathbb{N}$ and let $s_{x,n-1}$ be the line segment in K_{n-1} containing $f_{n-1}(x)$. We let:

$$l_n(x) = \begin{cases} l_{n-1}(x) * 0 & \text{if } f_n(x) \in B(s_{x,n-1}) \\ l_{n-1}(x) * 1 & \text{if } f_n(x) \in C(s_{x,n-1}) \\ l_{n-1}(x) & \qquad\qquad otherwise \end{cases}$$

Stages of labeling correspond to stages of Koch construction. If in the n-th stage of Koch construction x "stays in the same place" (*i.e.*, $f_n(x) = f_{n-1}(x)$), then the label for x at stage n remains what it was in the previous stage (*i.e.*, $l_n(x) = l_{n-1}(x)$). If on the other hand x gets "pushed up" to a side of an equilateral triangle introduced at stage n, then the new label $l_n(x)$ appends a 0 or 1 to the old label $l_{n-1}(x)$ (depending on which side of the equilateral triangle – *i.e.*, "left" or "right".)

Note that some elements in $[0, 1]$ "stabilize" over successive labelings and some do not. More precisely, some but not all points $x \in [0, 1]$ satisfy the following condition:

$$(*) \exists N \in \mathbb{N} \text{ such that } \forall n \geq N, l_n(x) = l_N(x)$$

If every point in the interval stabilized, we could happily restrict our attention to the infinite binary tree T_2 (*without* limits) and use this tree to label points in the real interval [0, 1]. The fact that many – in fact uncountably many – points do *not* stabilize is our motivation for passing from T_2 to T_2^+. Our final labeling function, l, agrees with stage-wise labeling functions on points that stabilize, but assigns limit nodes of T_2^+ to all points that do not stabilize. We define the function $l : [0, 1] \to \mathbb{T}_2^+$ as follows:

$$l(x) = \begin{cases} l_N(x) & \text{if } x \text{ satisfies } (*) \\ t & \text{otherwise} \end{cases}$$

where t is the unique countable sequence over $\{0, 1\}$ that has $l_n(x)$ as initial segment for each $n \in \mathbb{N}$.

To take a simple example, it is clear that the point $\frac{1}{3}$ stabilizes and therefore $l(\frac{1}{3})$ is a finite string. Indeed, $l(\frac{1}{3}) = \langle\cdot\rangle$, as $l_n(\frac{1}{3}) = l_0(\frac{1}{3}) = \langle\cdot\rangle$ for all $n \in \mathbb{N}$. Note that successive labeling functions, l_n, are monotonic in the following sense: For any $x \in [0, 1]$, if $m < n$, then $l_n(x)$ is an descendant of $l_n(x)$ (*i.e.*, $l_n(x) = l_m(x) * t$ for some $t \in \Sigma^+$). Moreover, $l(x)$ is a descendant of $l_n(x)$ for all $n \in \mathbb{N}$ (*i.e.*, $l(x) = l_n(x) * t_n$ for some $t_n \in \Sigma^+$).

Theorem 4.2 *$l : [0, 1] \to \mathbb{T}_2^+$ is a full, interior map*

The proof of this theorem is given in the section below. We state as corollaries the two main results of this paper:

Corollary 4.3 *$S4$ is complete for the class of models over the real interval $[0, 1]$.*

Proof. Immediate from Fact 3.8, Theorem 3.18, and Theorem 4.2.

Corollary 4.4 *$S4$ is complete for the class of models over Koch curve, K.*

Proof. By the map $l \circ f^{-1} : K \to \mathbb{T}_2^+$. That the composition is full-interior is immediate from Claim 4.1 and Corollary 4.3.

4.2 l is a Full Interior Map

In this section, we prove Theorem 4.2.

Proof.

As before, for any finite node $s \in \mathbb{T}_2^+$, let B_s be the basic open set $\{s * t \,|\, t \in \Sigma^+\}$.

1. (Continuous) Let U be a basic open set in $\mathbb{T}_2^+$. Then $U = B_s$ for some finite node $s \in \mathbb{T}_2^+$. Suppose $x \in l^{-1}(B_s)$. We show there is an open set $O \subseteq [0,1]$ such that $x \in O \subseteq l^{-1}(B_s)$. By construction of the functions l_n, there exists a least $N \in \mathbb{N}$ such that $l_N(x) = s$. Moreover, at stage N all points belonging to some open interval O which contains x are labeled by s – *i.e.*, for each $y \in O$, $l_N(y) = s$. By monotonicity of the labeling functions, $l(y)$ is a descendant of $l_N(y)(= s)$ for each $y \in O$. So $O \subseteq l^{-1}(B_s)$. Moreover, $x \in O$ and O is open, as needed.

2. (Open) We introduce the notion of a maximal, uniformly labeled (MUL) interval under l_n. In particular, $I \subseteq [0,1]$ is a MUL interval under l_n if for all $x, y \in I$, $l_n(x) = l_n(y)$, and there does not exist a strictly bigger interval $I' \supset I$ with this property. With slight abuse of notation, where I is a MUL interval under l_n, all of whose points are labeled by some node t, we will write $l_n(I) = t$. Note that for each point $x \in [0,1]$, x belongs to successively smaller MUL intervals under the finite labeling functions, $l_1, l_2, l_3, \ldots$. (Thus, *e.g.*, for $x = 1/4$, x belongs to the MUL interval $[0, \frac{1}{3}]$ under l_1, then to the MUL interval $[\frac{2}{9}, \frac{1}{3}]$ under l_2, etc.) Letting $I_{x,n}$ be the MUL interval under l_n containing x, we have that $\mathit{length}\,(I_{x,n}) \to_{n\to\infty} 0$. It follows that if $O \subseteq [0,1]$ is open, and $x \in O$, then for large enough n, $I_{x,n} \subseteq O$.

   Now let $O \subseteq [0,1]$ be open, and suppose $s \in l(O)$ – that is, $l(x) = s$ for some $x \in O$. We need to show that there exists an open set $U \subseteq \mathbb{T}_2^+$ such that $s \in U \subseteq l(O)$.

   If (case 1) s is finite, then for large enough n, $I_{x,n} \subseteq O$ and $l_n(I_{x,n}) = s$. We claim that $l(I_{x,n}) = B_s$. Since $I_{x,n} \subseteq O$, we have $s \in B_s \subseteq l(O)$, and B_s is open, as needed. (Proof of the claim: By monotonicity of the labeling functions, we know that $l(I_{x,n}) \subseteq B_s$. The difficult part is to show that $B_s \subseteq l(I_{x,n})$ – in particular, that every *limit node* in B_s labels some point in $I_{x,n}$ under l. We prove this part, and leave the case for finite nodes to the reader. Let r be a limit node in B_s. Then $r = s * r'$ for some countably infinite string $r' \in \Sigma^+$. We write $r' = (r'_1, r'_2, r'_3, \ldots)$. We need to find $x' \in I_{x,n}$ such that $l(x') = r$. It will be useful for us to label different segments of an MUL interval, I, by $A(I), B(I), C(I)$, and $D(I)$, just as we labeled different parts of the line segments in K_n above.[12] We now define a sequence of points $x_n \in [0,1]$, recursively. For the base step: If $r'_1 = 0$, then let x_1 be some point in $B(I_{x,n})$; if $r'_1 = 1$, then let x_1 be some point in $C(I_{x,n})$. For the recursive step, assume we have defined the points $x_1, \ldots, x_k$. Then if $r'_{k+1} = 0$, let x_{k+1} be some point in $B(I_{x_k,n+k})$; if $r'_{k+1} = 1$, then let x_{k+1} be some point in $C(I_{x_k,n+k})$. By construction, for each $k \in \mathbb{N}$, we have $x_{k+1}, x_k \in I_{x_k,n+k}$. So $|x_{k+1} - x_k| \leq \mathit{length}\,(I_{x_k,n+k}) \to_{k\to\infty} 0$. Thus the sequence $\{x_k\}$ is Cauchy, hence convergent. We let $x' = \lim_{k\to\infty} x_k$. It is then clear by construction that $x' \in I_{x,n}$ and $l(x') = s * r' = r$, as needed.)

   If (case 2) s is a limit node, then $l_n(I_{x,n})$ is a finite ancestor of s, for each $n \in \mathbb{N}$. We pick n large enough so that $I_{x,n} \subseteq O$ and let $t = l_n(I_{x,n})$. Then, as in the previous case,

[12] Thus, if $I = (i_1, i_2)$, we have:
$B(I) = (i_1 + \frac{1}{3}(i_2 - i_1), i_1 + \frac{i_2 - i_1}{2})$

$C(I) = (i_1 + \frac{i_2 - i_1}{2}, i_2 - \frac{1}{3}(i_2 - i_1))$

$l(I_{x,n}) = B_t$. Moreover, $s \in B_t$ by monotonicity of the labeling functions. Since $I_{x,n} \subseteq O$, we have $s \in B_t \subseteq l(O)$, and B_t is open, as needed.

3. (Surjective) We know already that for some $x \in [0,1]$, $l(x) = \langle \cdot \rangle$, which is the root of T_2^+ (pick, *e.g.*, $x = \frac{1}{3}$). Moreover, the entire interval [0, 1] is open. So by the fact that l is open, $l[0,1]$ is open, and contains the root of $\mathbb{T}_2^+$. Since every node in $\mathbb{T}_2^+$ is a descendant of the root, it follows that l is surjective.

This completes the proof of the theorem.

5 Conclusions and further directions

The results just proved show that fractal techniques can be usefully and relatively smoothly applied to problems in the topological semantics for the standard modal logic $S4$. In conclusion, we wish to suggest that the usefulness of fractal techniques is much more general. It extends to related intensional languages as long as they (*i*) have a topological interpretation in one of the standard metric topologies, and (*ii*) do not exceed a certain level of expressive power. As we argued in the introduction, the results of Section 4 and the techniques above are not tailor-made for proving completeness of $S4$ for the real line. Rather, these techniques should be seen as a recipe for obtaining completeness results for a larger variety of languages and with respect to the full range of Euclidean and other metric topologies.

The main technique is developed to relate formally the somewhat peculiar tree topologies (non-Hausdorff) with more familiar metric spaces. One simply finds *a suitable tree* for which the desired completeness result is easily proved, and then constructs fractal-based maps from the topology in question to the tree. For instance, for the proof of completeness of $S4$ for the rationals, this tree is the infinite binary tree, T_2; for completeness for the reals, as we have demonstrated above, the tree is $\mathbb{T}_2^+$ (the Wilson tree). Once the right tree is found, and completeness has been shown for the tree, one finds an appropriate fractal which facilitates topological completeness transfer from the tree to the desired metric space. In the case of $S4$ for the reals, the needed fractal was the Koch Curve. In the case of modal topological products logics over rationals, the appropriate fractal is known as Vicsek fractal.[13]

Topological concepts have a plethora of important applied uses: from issues related to our understanding of space and time, to dynamical systems. In logic, in our view, we ought to have a ready set of tools for exploring the logic of topological spaces, their inner structure, and the strength of languages needed to express them. The fractal techniques introduced here are a small step in the direction of accumulating such tools.

Several lines of inquiry extend the results and techniques of this paper. Our paper tentatively titled "Sierpinski Carpet, Menger Sponge, and Fractal Techniques in Topological Modal Logic" proves completeness of $S4$ for the plane $\mathbb{R}^2$ and the cube $\mathbb{R}^3$. These proofs are both direct: they do not rely on completeness for the real line. For this reason, the proofs provide a deeper understanding of the plane, the cube, and their topological and logical structure. We are also currently working on understanding the general case, $\mathbb{R}^n$, and the countably infinite case, $\mathbb{R}^\infty$.

[13] We have used this fractal in the main proof of [3]. In fact, the discovery that the main result there relied on a fractal technique is what led to this line of research.

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