What is Possibility Semantics?

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What is Possibility Semantics?

One sentence answer: Possibility Semantics is a generalization of Possible World Semantics, based on partial possibilities instead of complete possible worlds.

Outline for today (possibly continuing tomorrow):

1. Classical possibility semantics
2. Non-classical possibility semantics
3. Adding modalities
4. Historical notes.

For associated reading, see the course webpage.

First, what is Possible World Semantics?

For the purposes of this course, a possible worlds semantics is a semantics on which

- propositions are (or correspond to) sets, and
- ‘and’ and ‘or’ are interpreted as intersection and union.

Thus, not only standard relational semantics for classical modal logic but also relational semantics for *intuitionistic logic* count as possible world semantics, despite the fact that the latter does not interpret negation as set-theoretic complement.

It’s a broad definition but it rules out a lot.

To foreshadow a bit: note that intersection and union obey the distributive law, $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$, so possible world semantics in the above sense won’t be able to give semantics for logics without distributivity of ‘and’ and ‘or’—which will be very important when we turn to epistemic modals...

What is Possibility Semantics?

Now back to the question: what is Possibility Semantics?

Preliminaries

Partially ordered sets

First, we need a few definitions.

Suppose we have a set L of *propositions* and a relation $\leq$ of *entailment* between propositions, which we assume is a *partial order*:

- reflexive: $a \leq a$;
- transitive: $a \leq b$ and $b \leq c$, then $a \leq c$;
- antisymmetric: if $a \leq b$ and $b \leq a$, then $a = b$.

Anti-symmetry implies we are adopting a coarse-grained notion of proposition.

The pair $(L, \leq)$ is a partially ordered set or **poset**.

Lattices

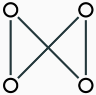
A poset $(L, \leq)$ is a **lattice** if every two-element set $\{a, b\} \subseteq L$ has a **greatest lower bound** (or **meet**), denoted $a \sqcap b$, and **least upper bound** (or **join**), denoted $a \sqcup b$:

- $a \sqcap b \leq a$ and $a \sqcap b \leq b$ (lower bound);
- if $c \leq a$ and $c \leq b$, then $c \leq a \sqcap b$ (greatest lower bound).

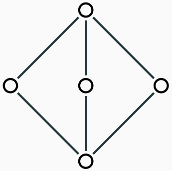
- $a \leq a \sqcup b$ and $b \leq a \sqcup b$ (upper bound bound);
- if $a \leq c$ and $b \leq c$, then $a \sqcup b \leq c$ (least upper bound).

Compare the introduction and elimination rules for ‘and’ and ‘or’.

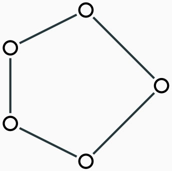
Examples: not lattices



Examples: lattices

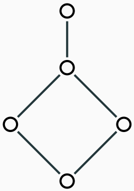


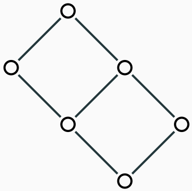
M_3

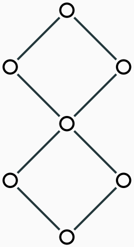


N_5

Examples: lattices







Complete lattices

A poset $(L, \leq)$ is a **complete lattice** if every collection $\{A_i\}_{i \in I} \subseteq L$ has a **greatest lower bound** (or **meet**), denoted $\bigsqcap_{i \in I} A_i$ or $\bigsqcap \{A_i \mid i \in I\}$:

- $\bigsqcap_{i \in I} A_i \leq A_j$ for all $j \in I$ (lower bound);
- if $B \leq A_j$ for all $j \in I$, then $B \leq \bigsqcap_{i \in I} A_i$ (greatest lower bound).

It follows that each $\{A_i\}_{i \in I}$ also has a **least upper bound** (or **join**) (definable as $\bigsqcap \{a \in L \mid a \text{ an upper bound of } \{A_i\}_{i \in I}\}$), denoted $\bigsqcup_{i \in I} A_i$ or $\bigsqcup \{A_i \mid i \in I\}$:

- $A_j \leq \bigsqcup_{i \in I} A_i$ for all $j \in I$ (upper bound bound);
- if $A_j \leq B$ for all $j \in I$, then $\bigsqcup_{i \in I} A_i \leq B$ (least upper bound).

An easy inductive proof shows that any finite lattice is a complete lattice.

Closure operators

We just need one more definition.

Given a nonempty set X, a *closure operator* on the powerset $\wp(X)$ of X is a function $c : \wp(X) \to \wp(X)$ that is:

- inflationary: $A \subseteq c(A)$;
- idempotent: $c(c(A)) = c(A)$;
- monotone: if $A \subseteq B$, then $c(A) \subseteq c(B)$.

If you haven't seen these before in topology, you've seen them in logic:
let $c(\Gamma)$ be the set of logical consequences of the set Γ of sentences.

Fixpoints of a closure operator

I have come to label some semantics a **possibility semantics** if it is based on interpreting 'and' and 'or' using the following classic result from lattice theory.

Proposition

Let X be a nonempty set and c a closure operator on $\wp(X)$. Then the fixpoints of c, i.e., those $A \subseteq X$ with $c(A) = A$, ordered by $\subseteq$ form a complete lattice with

$$\bigsqcap_{i \in I} A_i = \bigcap_{i \in I} A_i \text{ and } \bigsqcup_{i \in I} A_i = c(\bigcup_{i \in I} A_i).$$

Key: 'or' is interpreted as **closure of union** instead of union as in possible world semantics. Thus, a state $x \in X$ can settle a disjunction as true *without settling which disjuncts are true*. In this sense, x may be a *partial* state—what we call a 'possibility'.

Proposition

Let X be a nonempty set and c a closure operator on $\wp(X)$. Then the fixpoints of c, i.e., those $A \subseteq X$ with $c(A) = A$, ordered by $\subseteq$ form a complete lattice with

$$\prod_{i \in I} A_i = \bigcap_{i \in I} A_i \text{ and } \bigsqcup_{i \in I} A_i = c(\bigcup_{i \in I} A_i).$$

Proof. First, by monotonicity

$$c(\bigcap_{i \in I} A_i) \subseteq c(A_i) = A_i \text{ for each } i \in I \text{, so } c(\bigcap_{i \in I} A_i) \subseteq \bigcap_{i \in I} A_i.$$

By inflationarity, the converse inclusion also holds, so $\bigcap_{i \in I} A_i$ is a fixpoint. Clearly $\bigcap_{i \in I} A_i$ is the greatest lower bound of $\{A_i\}_{i \in I}$. Now $c(\bigcup_{i \in I} A_i)$ is a fixpoint by idempotence and is an upper bound of $\{A_i\}_{i \in I}$ by inflationarity. To see that it is the *least* upper bound, observe that if B is an upper bound, then by monotonicity,

$$\bigcup_{i \in I} A_i \subseteq B \Rightarrow c(\bigcup_{i \in I} A_i) \subseteq c(B) = B.$$

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Possibility semantics for classical logic

Possibilities partially ordered by refinement

Consider a poset $(X, \sqsubseteq)$. We call X a set of *possibilities* and $\sqsubseteq$ the relation of *refinement*. Take $x \sqsubseteq y$ to mean that x is a *refinement* or *further specification* of y.

Given $A \subseteq X$, let ${\downarrow}A = \{x \in X \mid \exists y \in A : x \sqsubseteq y\}$. For $x \in X$, let ${\downarrow}x = {\downarrow}\{x\}$.

Exercise

$\downarrow$ is a closure operator on $\wp(X)$.

A *downset* is a fixpoint of $\downarrow$: ${\downarrow}A = A$. Propositions should be at least downsets.

Regular open sets

But $\downarrow$ is not the closure operator we want. We want $\rho : \wp(X) \to \wp(X)$ defined by

$$\rho(A) = \{x \in X \mid \forall x' \sqsubseteq x \, \exists x'' \sqsubseteq x' : x'' \in \downarrow A\}.$$

A *regular open set* is a fixpoint of ρ: $\rho(A) = A$.

Exercise

A set $A \subseteq X$ is regular open iff it satisfies the following conditions for all $x, x' \in X$:

1. **persistence**: if $x \in A$ and $x' \sqsubseteq x$, then $x' \in A$;
2. **refinability**: if $x \notin A$, then $\exists x' \sqsubseteq x \, \forall x'' \sqsubseteq x'$: $x'' \notin A$.

Persistence says: if x settles A as true, so does any refinement of x.

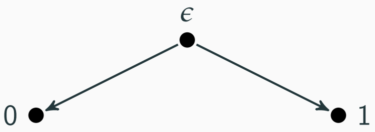
Refinability says: if x does not settle A as true, then x can be refined to some x' that settles A as *false*, so no refinement x'' of x' settles A as true.

Finite example

Let's start with an example of a *finite* poset. Although possible world semantics can already represent any finite Boolean algebra (as the powerset of its set of atoms—more on this shortly), still we'll be able to do interesting things with finite posets.

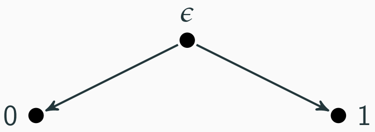
Finite example

When we draw posets, an arrow from y to x means $x \sqsubseteq y$ (arrows point to refinements); we don't draw reflexive arrows or arrows implied by transitivity.



In classical possibility semantics, a **world** is a possibility that is refined only by itself. So we have two worlds: 0 and 1.

Finite example



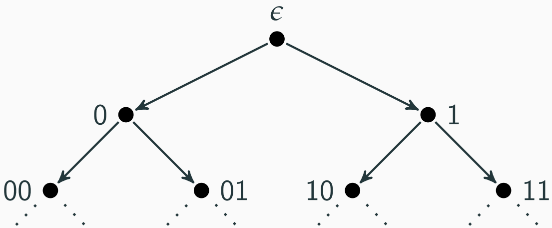
The regular open sets are $\varnothing$, $\{0\}$, $\{1\}$, and $\{\epsilon, 0, 1\}$.

Note that $\{\epsilon\}$, $\{\epsilon, 0\}$, and $\{\epsilon, 1\}$ are not regular open, since they violate **persistence**.

And $\{0, 1\}$ is not regular open since it violates **refinability**: $\epsilon \notin \{0, 1\}$, but there's no refinement of ϵ all of whose refinements are not in $\{0, 1\}$.

Note that $\{0\} \sqcup \{1\} = \{\epsilon, 0, 1\}$, so $\sqcup$ is not union!

Infinite example

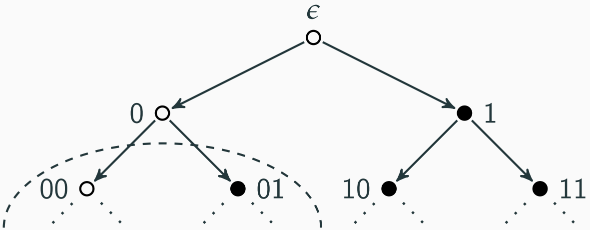


Every principal downset $\downarrow x$ is regular open: persistence is immediate, and for refinability, if $y \notin \downarrow x$, then there is a child $y' \sqsubseteq y$ such that for all $y'' \sqsubseteq y'$, $y'' \notin \downarrow x$.

$\downarrow x \cup \downarrow x'$ is also regular open provided x and x' are not children of the same node.

Infinite example

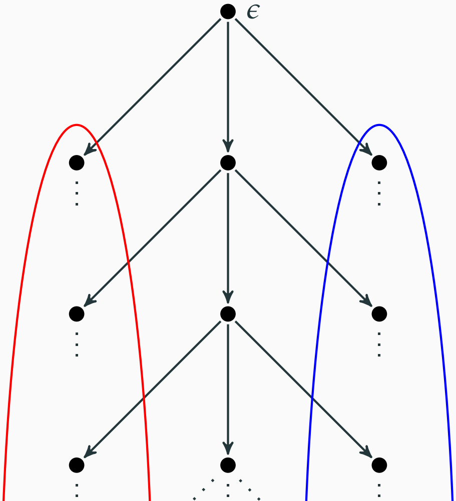
As in our first example, $\downarrow 00 \cup \downarrow 01$ is not regular open: for $0 \notin \downarrow 00 \cup \downarrow 01$, yet there is no $y \sqsubseteq 0$ such that for all $z \sqsubseteq y$, $z \notin \downarrow 00 \cup \downarrow 01$, so refinability fails for $\downarrow 00 \cup \downarrow 01$.



Another set that is not regular open is $U = \{\sigma \in X \mid \sigma \text{ contains at least one } 1\}$, represented by the filled-in black nodes in the diagram above: for $\epsilon \notin U$, yet there is no $y \sqsubseteq \epsilon$ such that for all $z \sqsubseteq y$, $z \notin U$, so refinability fails for U.

Another infinite example

In the following poset, the sets outlined red and blue are each regular open, but their union is *not* regular open. Also note that $\epsilon \in \mathit{Red} \sqcup \mathit{Blue}$.



Regularization is a closure operator

We want $\rho : \wp(X) \to \wp(X)$ defined by

$$\rho(A) = \{x \in X \mid \forall x' \sqsubseteq x \, \exists x'' \sqsubseteq x' : x'' \in {\downarrow}A\}.$$

Exercise

ρ is a closure operator on $\wp(X)$. Moreover, for any downsets A and B, it is also

- multiplicative: $\rho(A) \cap \rho(B) = \rho(A \cap B)$ (the $\supseteq$ direction is just monotonicity).

The complete lattice $\mathcal{RO}(X, \sqsubseteq)$

Exercise

For any poset $(X, \sqsubseteq)$, ρ is a closure operator on $\wp(X)$.

Proposition

Let X be a nonempty set and c a closure operator on $\wp(X)$. Then the fixpoints of c, i.e., those $A \subseteq X$ with $c(A) = A$, ordered by $\subseteq$ form a complete lattice with

$$\prod_{i \in I} A_i = \bigcap_{i \in I} A_i \text{ and } \bigsqcup_{i \in I} A_i = c(\bigcup_{i \in I} A_i).$$

Thus, we know the collection $\mathcal{RO}(X, \sqsubseteq)$ of regular open sets ordered by $\subseteq$ forms a complete lattice. Notice how the join is calculated:

$$\bigsqcup_{i \in I} A_i = \rho(\bigcup_{i \in I} A_i) = \{x \in X \mid \forall x' \sqsubseteq x' \, \exists x'' \sqsubseteq x' \, \exists i \in I : x'' \in A_i\}.$$

Negation on $\mathcal{RO}(X, \sqsubseteq)$

Moreover, there is a natural negation operation $\neg$ on $\mathcal{RO}(X, \sqsubseteq)$ defined by:

$$\neg A = \{x \in X \mid \forall x' \sqsubseteq x \; x' \notin A\}.$$

Note that we can then rewrite **Refinability** as: if $x \notin A$, then $\exists x' \sqsubseteq x$: $x' \in \neg A$.

Exercise

For any downset A, we have $\rho(A) = \neg\neg A$.

Now what can we say about the properties of the complete lattice $\mathcal{RO}(X, \sqsubseteq)$?

Boolean algebras

A lattice $(L, \leq)$ is **bounded** if it has a greatest element with respect to $\leq$, denoted 1 (or $\top$), and a least element with respect to $\leq$, denoted 0 (or $\bot$).

Every complete lattice is bounded, since the least upper bound of $\varnothing$ is 0 and the greatest lower bound of $\varnothing$ is 1.

A bounded lattice is **complemented** if for every $a \in L$ there is an element $\neg a$, called a *complement of* a, such that $a \sqcup \neg a = 1$ and $a \sqcap \neg a = 0$.

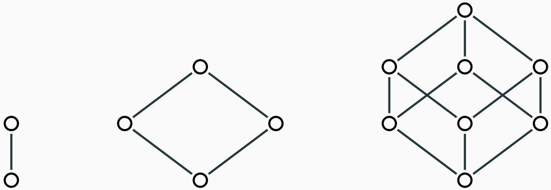
A lattice is **distributive** if for all $a, b, c \in L$, we have $a \sqcap (b \sqcup c) = (a \sqcap b) \sqcup (a \sqcap c)$.

Exercise

A lattice is distributive iff for all $a, b, c \in L$, we have $a \sqcup (b \sqcap c) = (a \sqcup b) \sqcap (a \sqcup c)$.

A **Boolean algebra** is a complemented distributive lattice.

Examples: Boolean algebras



The complete Boolean algebra $\mathcal{RO}(X, \sqsubseteq)$

Theorem (Tarski 1937)

For any poset $(X, \sqsubseteq)$, $\mathcal{RO}(X, \sqsubseteq)$ *is a complete Boolean algebra with*

$$\begin{aligned}
\neg A &= \{x \in X \mid \forall x' \sqsubseteq x \; x' \notin A\} \\
\bigsqcap_{i \in I} A_i &= \bigcap_{i \in I} A_i \\
\bigsqcup_{i \in I} A_i &= \rho\Big(\bigcup_{i \in I} A_i\Big) = \{x \in X \mid \forall x' \sqsubseteq x \, \exists x'' \sqsubseteq x' \, \exists i \in I : x'' \in A_i\}.
\end{aligned}$$

We already know $\mathcal{RO}(X, \sqsubseteq)$ is a complete lattice. And it's easy to check that $\neg A$ is a complement of A. Then **multiplicativity** of ρ gives us distributivity of the lattice:

$$\begin{aligned}
A \sqcap (B \sqcup C) &= \rho(A) \cap \rho(B \cup C) \\
&= \rho(A \cap (B \cup C)) = \rho((A \cap B) \cup (A \cap C)) = (A \sqcap B) \sqcup (A \sqcap C).
\end{aligned}$$

Key ideas

Key idea of classical possibility semantics: propositions belong to the Boolean algebra of regular open sets of a poset.

Key idea of classical possible world semantics: propositions belong to the Boolean algebra of all subsets of a set.

What's the difference so far?

Representation of complete Boolean algebras

Theorem

Each complete Boolean algebra $(B, \leq)$ *is isomorphic to* $\mathcal{RO}(B_+, \leq_+)$*, where* B_+ *is the set of nonzero elements of* B *and* $\leq_+$ *is the restriction of* $\leq$ *to* B_+*, via the map* $a \mapsto \{b \in B_+ \mid b \leq a\}$.

Let us contrast this with what possible world semantics provides. An **atom** in a Boolean algebra $(B, \leq)$ is an $a \in B$ such that $0 < a$ and there is no b with $0 < b < a$; and the algebra is **atomic** if for each $b \in B$, there is an atom $a \leq b$.

Theorem (Tarski 1935)

For any set X*, the poset* $(\wp(X), \subseteq)$ *is a complete and atomic Boolean algebra (CABA) in which complement, meet, and join are given by set-theoretic complement, intersection, and union. Conversely, each CABA* B *is isomorphic to* $(\wp(At(B)), \subseteq)$*, where* $At(B)$ *is the set of atoms of* B*, via the map* $b \mapsto \{a \in At(B) \mid a \leq b\}$.

Representation of arbitrary Boolean algebras

To represent arbitrary (possibly incomplete) Boolean algebras, we equip a poset with a distinguished subalgebra of $\mathcal{RO}(X, \sqsubseteq)$; so not all regular opens count as propositions.

Definition

A *general refinement frame* is a triple $\mathcal{F} = (X, \sqsubseteq, \mathcal{P})$ where $(X, \sqsubseteq)$ is a poset and $\mathcal{P} \subseteq \mathcal{RO}(X, \sqsubseteq)$ is closed under binary intersection and the $\neg$ in $\mathcal{RO}(X, \sqsubseteq)$.

To go from a BA to a possibility frame, we can construct possibilities as *proper filters* (which are like consistent but not necessarily maximally-consistent theories)...

Representation of arbitrary Boolean algebras

A *filter* in a Boolean algebra $(B, \leq)$ is a nonempty $F \subseteq B$ that is closed under $\sqcap$ ($a, b \in F$ implies $a \sqcap b \in F$) and upward closed under $\leq$; the filter is *proper* if $F \neq B$.

Theorem

For any Boolean algebra $(B, \leq)$, the triple $(X, \sqsubseteq, \mathcal{P})$ where

- *X is the set of proper filters of $(B, \leq)$,*
- *$F \sqsubseteq G$ iff $F \supseteq G$, and*
- *$\mathcal{P} = \{\widehat{a} \mid a \in B\}$ where $\widehat{a} = \{F \in X \mid a \in F\}$*

is a general refinement frame such that $(B, \leq)$ is isomorphic to $(\mathcal{P}, \subseteq)$.

Unlike Stone's representation of arbitrary Boolean algebras as algebras of sets, the above theorem can be proved without (even weak fragments of) the axiom of choice (see our "Choice-Free Stone Duality" for further developments of this point).

Possibility models for classical propositional logic

Let $\mathcal{L}$ be the usual language of propositional logic based on a set Prop of variables.

A **possibility model** for $\mathcal{L}$ is a triple $\mathcal{M} = (X, \sqsubseteq, V)$ where $(X, \sqsubseteq)$ is a poset and $V : \text{Prop} \to \mathcal{RO}(X, \sqsubseteq)$. We then define the forcing relation $\Vdash$ recursively as follows:

- $\mathcal{M}, x \Vdash p$ iff $x \in V(p)$;
- $\mathcal{M}, x \Vdash \neg\varphi$ iff $\forall x' \sqsubseteq x\ \mathcal{M}, x' \nVdash \varphi$;
- $\mathcal{M}, x \Vdash \varphi \wedge \psi$ iff $\mathcal{M}, x \Vdash \varphi$ and $\mathcal{M}, x \Vdash \psi$.
- $\mathcal{M}, x \Vdash \varphi \vee \psi$ iff $\forall x' \sqsubseteq x\ \exists x'' \sqsubseteq x'$: $\mathcal{M}, x'' \Vdash \varphi$ or $\mathcal{M}, x'' \Vdash \psi$.

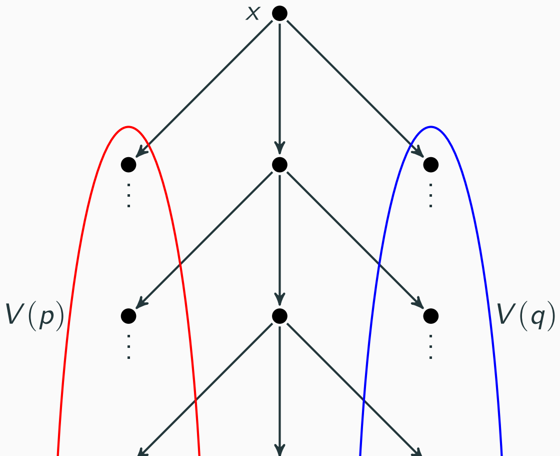
Exercise

Defining $\varphi \to \psi$ as $\neg\varphi \vee \psi$, we have:

- $\mathcal{M}, x \Vdash \varphi \to \psi$ iff $\forall x' \sqsubseteq x$ if $\mathcal{M}, x' \Vdash \varphi$, then $\mathcal{M}, x' \Vdash \psi$.

Example

Let $V(p)$ be the set outlined in red and $V(q)$ the one in blue.
Then $\mathcal{M}, x \Vdash p \vee q$, $\mathcal{M}, x \Vdash p \to \neg q$, and $\mathcal{M}, x \Vdash q \to \neg p$.



Possibility models for classical propositional logic

Define $\Gamma \vDash \varphi$ iff for all possibilities models $\mathcal{M}$ and possibilities x in $\mathcal{M}$,

if $\mathcal{M}, x \Vdash \gamma$ for all $\gamma \in \Gamma$, then $\mathcal{M}, x \Vdash \varphi$.

Proposition (Soundness and Completeness)

$\Gamma \vDash \varphi$ iff φ is derivable from Γ in classical propositional logic.

Soundness (right to left) uses Tarski's theorem that $\mathcal{RO}(X, \sqsubseteq)$ is a Boolean algebra. Completeness (left to right) uses completeness with respect to the standard semantics for classical propositional logic, which is equivalent to possibility semantics using possibility models containing only one possibility!

Possibility models for classical propositional logic

If our only goal were to give semantics for classical propositional logic, then of course there would be no point in using possibility semantics based on posets.

But we want to add modalities, propositional quantifiers, first-order quantifiers, etc., and then there will be a point!

Possibility semantics for non-classical logic

Approaches to non-classicality

There are at least three approaches to non-classical possibility semantics, which are compared in "Three roads to complete lattices."

Guillaume Massas has extensively developed one of the roads—based on having two refinement-like relations instead of one—in his "B-frame duality."

Here I'll discuss another road, which switches from refinement to *compatibility* as in "Compatibility and accessibility" and "The Orthologic of Epistemic Modals," also called *openness* in "A Fundamental Non-Classical Logic."

Basic notions

We begin with a pair $(X, \lhd)$ of a nonempty set X of **states** and a binary relation $\lhd$. We read $x \lhd y$ as x **is open to** y.

Consider the distinction between **accepting** a proposition and **rejecting** it:

- We want to allow for **partial** states that are completely noncommittal about a proposition, so *non-acceptance* of a proposition should not entail *rejection* of it.
- Moreover, we want to allow for states that reject a proposition without accepting the negation of it; for example, an intuitionist might *reject* a certain instance of the law of excluded middle, $A \vee \neg A$, but will certainly not accept its negation, $\neg(A \vee \neg A)$, which is an intuitionistic contradiction.

These notions can be linked with our notion of **openness** as follows:

- x **is open to** y iff x does not reject any proposition that y accepts.

Basic notions

Now if we start with $(X, \lhd)$ and a *proposition* (defined shortly) $A \subseteq X$, say that:

- x accepts A if $x \in A$;
- x rejects A if for all y such that $x \lhd y$, $y \notin A$;
- x accepts $\neg A$ if for all $y \lhd x$, $y \notin A$.

Then we will have that $x \lhd y$ iff x does not reject any proposition that y accepts.

Another result of the partiality of states is that accepting a disjunction does not require accepting either disjunct. Instead, x accepting $A \vee B$ will amount to:

no state open to x rejects both disjuncts.

What is a proposition?

In this setup, what is a proposition?

In possible world semantics, a proposition is an arbitrary subset of the set W of worlds.

But again, for us, only special subsets of X are propositions. . .

Propositions

Key idea: if x does not accept A, there should be some y open to x that rejects A:

$$\text{if } x \notin A, \text{ then } \exists y \lhd x \; \forall z \rhd y \; z \notin A.$$

Call a set A a **proposition** if it satisfies the condition for all $x \in X$.

Equivalently, we can define propositions as the fixpoints of a closure operator.

Define $c_{\lhd} : \wp(X) \to \wp(X)$:

$$c_{\lhd}(A) = \{x \in X \mid \forall x' \lhd x \; \exists x'' \rhd x' : x'' \in A\}.$$

So x is in $c_{\lhd}(A)$ iff every state open to x is open to some state in A.

Exercise

$c_{\lhd}$ is a closure operator on $\wp(X)$.

Hence the $c_{\lhd}$-fixpoints, those A with $c_{\lhd}(A) = A$, form a complete lattice as before.

Propositions

Proposition

Given any relational frame $(X, \lhd)$, the propositions ordered by the subset relation $\subseteq$ form a complete lattice in which

$$\begin{aligned} A \wedge B &= A \cap B \\ A \vee B &= \{x \in X \mid \forall y \lhd x \, \exists z \rhd y : z \in A \cup B\}. \end{aligned}$$

Moreover, the negation operation defined by

$$\neg_{\lhd} A = \{x \in X \mid \forall y \lhd x, y \notin A\}$$

satisfies the following:

1. $\neg_{\lhd} X$ is the minimum element of the lattice;
2. if $A \subseteq B$, then $\neg_{\lhd} B \subseteq \neg_{\lhd} A$.

Representation theorem

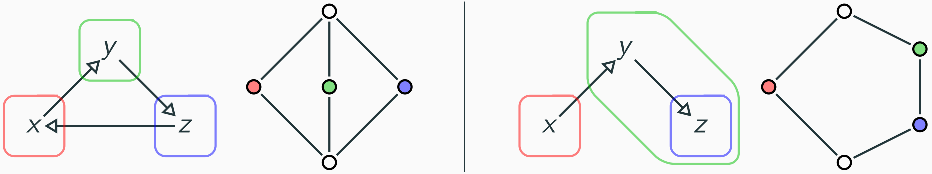
Theorem

Let L be a complete lattice with maximum element 1 and minimum element 0.

Let $\neg$ be a unary operation on L such that $\neg 1 = 0$ and if $a \leq b$, then $\neg b \leq \neg a$.

Then there is a relational frame $(X, \lhd)$ such that $(L, \neg)$ is isomorphic to the lattice of propositions with negation arising from $(X, \lhd)$ as in the previous results.

For proofs, see "A Fundamental Non-Classical Logic."



An arrow from w to v means $w \rhd v$. Reflexive loops are assumed but not shown.

One can check that $A \subseteq X$ is a proposition by checking that

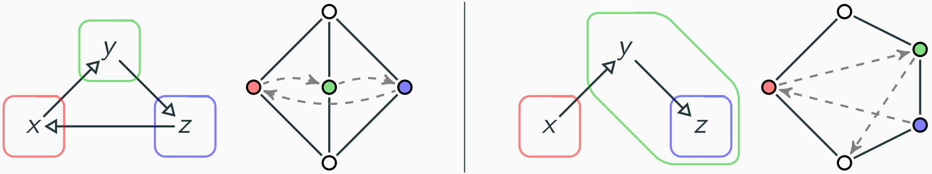
- from any $x \in X \setminus A$, you can see a state that cannot be seen from A.

In the **three-cycle** on the left, $\{y\}$ is a proposition because z and x can both see x, which cannot be seen from $\{y\}$. Yet $\{y, z\}$ is not a proposition, because x cannot see a state that cannot be seen from $\{y, z\}$, since both x and y can be seen from $\{y, z\}$.

In the acyclic (ignoring loops) but **non-transitive frame** on the right: $\{y\}$ is *not* a proposition, since now z cannot see a state that cannot be seen from $\{y\}$; but $\{y, z\}$ *is* a proposition, since x can see a state, namely x, that cannot be seen from $\{y, z\}$.

Negation

The $\neg_{\triangleleft}$ operation on the lattice of propositions is indicated by the dashed arrows (omitting the dashed arrows representing that $\neg_{\triangleleft}1 = 0$ and $\neg_{\triangleleft}0 = 1$):



From representation to semantics

A **relational model** is a triple $\mathcal{M} = (X, \lhd, V)$ where $(X, \lhd)$ is a relational frame and V maps each $p \in \mathrm{Prop}$ to a proposition $V(p) \subseteq X$.

We define a forcing relation between elements of X and formulas as follows:

1. $\mathcal{M}, x \Vdash p$ iff $x \in V(p)$;
2. $\mathcal{M}, x \Vdash \neg\varphi$ iff for all $x' \lhd x$, $\mathcal{M}, x' \nVdash \varphi$;
3. $\mathcal{M}, x \Vdash \varphi \wedge \psi$ iff $\mathcal{M}, x \Vdash \varphi$ and $\mathcal{M}, x \Vdash \psi$;
4. $\mathcal{M}, x \Vdash \varphi \vee \psi$ iff $\forall x' \lhd x\ \exists x'' \rhd x'$: $\mathcal{M}, x'' \Vdash \varphi$ or $\mathcal{M}, x'' \Vdash \psi$.

Given a class $\mathbb{C}$ of relational frames, we define $\varphi \vDash_{\mathbb{C}} \psi$ if for all $(X, \lhd) \in \mathbb{C}$, all models $\mathcal{M}$ based on $(X, \lhd)$, and all $x \in X$, if $\mathcal{M}, x \Vdash \varphi$, then $\mathcal{M}, x \Vdash \psi$.

Other semantics as special cases

The foregoing approach using frames $(X, \lhd)$ subsumes all of the following:

1. Classical possible world semantics: X is the set of worlds, $\lhd$ is identity;
2. Intuitionistic Kripke semantics: X is the set of states, $\lhd$ is the preorder relation;
3. Possibility semantics for classical logic: X is the set of possibilities, and $\lhd$ is defined from the refinement relation $\sqsubseteq$ as follows:

$$x \lhd y \text{ iff there is a } z \in X\text{: } z \sqsubseteq x \text{ and } z \sqsubseteq y;$$

4. Birkhoff/Goldblatt semantics for orthologic: $\lhd$ is reflexive and symmetric.

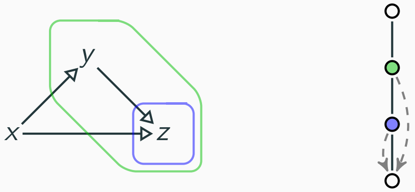


Figure 1: A relational frame realizing a Heyting algebra.

Looking at a diagram of a relational frame, one can check that $c_{\triangleleft}(A) = A$ by checking that the following holds:

- from any $x \in X \setminus A$, you can step forward along an arrow to a state x' that cannot step backward along an arrow into A.

Informally, "from x you can see a state that cannot be seen from A."

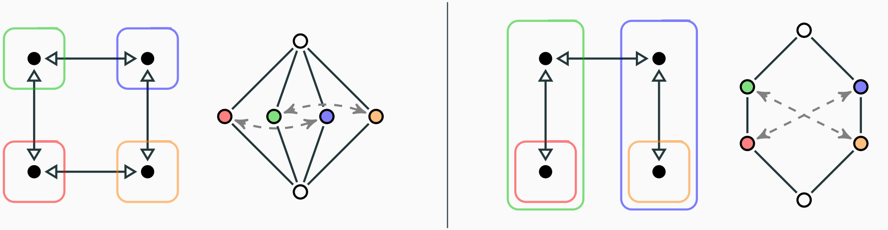


Figure 2: Relational frames realizing ortholattices.

Looking at a diagram of a relational frame, one can check that $c_{\lhd}(A) = A$ by checking that the following holds:

- from any $x \in X \setminus A$, you can step forward along an arrow to a state x' that cannot step backward along an arrow into A.

Informally, "from x you can see a state that cannot be seen from A."

Refinement and compossibility derived from openness

Definition

Given a relational frame $(X, \lhd)$ and $x, y \in X$:

1. x *pre-refines* y, written $x \sqsubseteq_{pr} y$, if for all $z \in X$, $z \lhd x$ implies $z \lhd y$;
2. x *post-refines* y, written $x \sqsubseteq_{po} y$, if for all $z \in X$, $x \lhd z$ implies $y \lhd z$;
3. x *refines* y, written $x \sqsubseteq y$, if x pre-refines and post-refines y;
4. x is *compossible with* y if there is a non-absurd $w \in X$ (where w is non-absurd if there is some $v \lhd w$) that refines x and pre-refines y.

Note that if $\lhd$ is symmetric, then pre-refinement and post-refinement are equivalent, and x is compossible with y just in case they have a common refinement.

Definition

A *compossible relational frame* is a relational frame $(X, \lhd)$ in which for any $x, y \in X$, if $x \lhd y$, then x is compossible with y.

Characterizations of varieties of lattices

Theorem

Let $(L, \leq)$ be a poset.

1. $(L, \leq)$ *is a complete lattice iff it is isomorphic to the* $c_{\lhd}$*-fixpoints of a (reflexive) relational frame ordered by* $\subseteq$.
2. $(L, \leq)$ *is an ortholattice iff it is isomorphic to the* $c_{\lhd}$*-fixpoints of a* **reflexive** *and* **symmetric** *relational frame.*
3. $(L, \leq)$ *is a complete Heyting algebra iff it is isomorphic to the* $c_{\lhd}$*-fixpoints of a* **reflexive** *and* **compossible** *relational frame ordered by* $\subseteq$.
4. $(L, \leq)$ *is a complete Boolean algebra iff it is isomorphic to the* $c_{\lhd}$*-fixpoints of a* **reflexive**, **symmetric**, *and* **compossible** *relational frame.*

For a proof, see "Compatibility and accessibility."

Representation of arbitrary lattices

Arbitrary lattices can be represented by equipping a relational frame with a distinguished subset $\mathcal{P}$ of $c_{\lhd}$-fixpoints closed under meet and join, resulting in what could be called a *general* relational frame $(X, \lhd, \mathcal{P})$.

See § 4 of "Compatibility and accessibility" and § 4.3 of "A Fundamental Non-Classical Logic."

Fundamental Logic

Fundamental Logic is a sublogic of intuitionistic logic and orthologic based only on the introduction and elimination rules for the logic connectives in Fitch-style natural deduction, studied in "A Fundamental Non-Classical Logic."

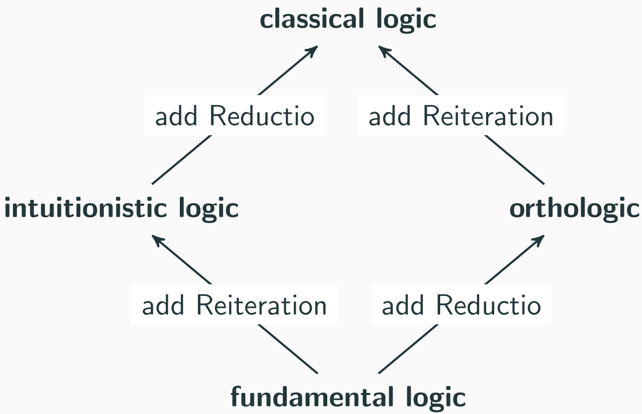
Intuitionistic Logic is obtained from FL by adding the rule Fitch called Reiteration.

Orthologic is obtained from FL by adding the rule of Reductio Ad Absurdum.

Fundamental logic is sound and complete with respect to the class of relational frames in which $\lhd$ satisfies reflexivity and

- **pseudosymmetry**: if $y \lhd x$, then there is a $z \lhd y$ such that z pre-refines x.

This is equivalent to: for every proposition A, if y accepts $\neg A$, then y rejects A.


classical logic
add Reductio
add Reiteration
intuitionistic logic
orthologic
add Reiteration
add Reductio
fundamental logic

Adding modalities

Refinement/openness and accessibility

Given a general refinement frame $(X, \sqsubseteq, \mathcal{P})$ or relational frame $(X, \lhd, \mathcal{P})$, we can give semantics for a normal $\Box$ by adding a binary relation R on X and requiring that

$$A \in \mathcal{P} \Rightarrow \Box_R A \in \mathcal{P}$$

where

$$\Box_R(A) = \{x \in X \mid R(x) \subseteq A\}$$

and $R(x) = \{y \in X \mid xRy\}$.

That $A \in \mathcal{P} \Rightarrow \Box_R A \in \mathcal{P}$ depends on the interaction of R with $\sqsubseteq$ (resp. $\lhd$).

Refinement and accessibility

For refinement, we can assume without loss of generality these interaction conditions:

- R-monotonicity: if $x' \sqsubseteq x$, $R(x') \subseteq R(x)$;
- R-regularity: $R(x) \in \mathcal{RO}(X, \sqsubseteq)$;
- R-refinability: if $y \in R(x)$, then $\exists x' \sqsubseteq x\ \forall x'' \sqsubseteq x'\ \exists y' \sqsubseteq y$: $y' \in R(x'')$.

These conditions guarantee that if $A \in \mathcal{RO}(X, \sqsubseteq)$, then $\Box_R A \in \mathcal{RO}(X, \sqsubseteq)$.

Note: the original paper on possibility semantics for modal logic, Humberstone's (1981) "From Worlds to Possibilities" had a stronger version of R-refinability, but this turned out to be too strong for the general theory, so we weakened it to R-refinability above in § 2.3 of "Possibility Frames and Forcing for Modal Logic."

Openness and accessibility

For openness, we can assume the following without loss of generality:

- $z \lhd_R x$, then $\exists x' \lhd x \; \forall x'' \rhd x' \;\; z \lhd_R x''$,

where $z \lhd_R x$ is an abbreviation for $\exists y : z \lhd y \in R(x)$.

This condition (from Prop. 4.5 of "Compatibility and accessibility") guarantees that if $A = c_{\lhd}(A)$, then $\Box_R A = c_{\lhd}(\Box_R A)$.

We will see concrete examples of refinement frames and relational frames equipped with accessibility later in this course.

Historical notes

Historical sources

Our starting point—that fixpoints of a closure operator on a powerset form a complete lattice—goes back to E. H. Moore (1910) and G. Birkhoff in the 1930s.

As we noted, classical possibility semantics can be traced back to Tarski (1935, 1937). The regular open algebra of a poset is used extensively in forcing in set theory.

Classical first-order possibility semantics dates to Fine (1971) and van Benthem (1981).

The addition of modal accessibility to possibility semantics goes back to Humberstone's (1981) "From Worlds to Possibilities," which coined the term 'possibility semantics'.

Non-classical possibility semantics using relational frames can be traced back to Miroslav Ploščica's paper, "A natural representation of bounded lattices," *Tatra Mountains Mathematical Publication*, Vol. 5 (1995), pp. 75-88.

Next up

Possibility semantics for **epistemic modals**. . .