Possibilities for Epistemic Modals

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ESSLLI 2023

Modal logic and formal semantics of natural language

Today we'll discuss an application of possibility semantics to the formal semantics of natural language: my paper with Matthew Mandelkern (NYU),

"The Orthologic of Epistemic Modals" (arXiv:2203.02872),

concerning the epistemic modals 'might' and 'must' in natural language.

Modal logic and formal semantics of natural language

A modest proposal about logic and formal semantics of natural language:

> Axiomatization can . . . be seen as a way of systematically and perspicuously revealing what the entailment predictions of a given formal semantics actually are. [W]e would like to argue that such an activity can indeed be valuable in the search for adequate accounts of natural language meaning.
>
> (Holliday and Icard, "Axiomatization in the Meaning Sciences," in *The Science of Meaning: Essays on the Metatheory of Natural Language Semantics*, Oxford, 2018)

Outline

- ‘Might’ and ‘must’ in natural language
- Algebraic semantics
- Possibility semantics
- Constructing possibilities from worlds
- Probability and conditionals (time permitting)

Natural language

Wittgenstein sentences

Wittgenstein sentences are sentences of the following forms, where $\Diamond$ is 'might':[1]

- $p \wedge \Diamond\neg p$
- $\neg p \wedge \Diamond p$
- $\Diamond p \wedge \neg p$
- $\Diamond\neg p \wedge p$.

[1] Yalcin (2007) calls the right-modal versions 'epistemic contradictions'.

Wittgenstein sentences

Wittgenstein sentences are *unassertable*:

(1) #Sue is sick and she might not be.

(2) #Sue might be sick, but she isn't.

But is the explanation semantic or pragmatic? Compare *Moore sentences*:

(3) #Sue is sick and I don't know it.

Wittgenstein sentences

The explanation is semantic. For Wittgenstein sentences embed like contradictions, unlike Moore sentences.[2]

[2] Groenendijk, Stokhof, and Veltman (1996); Aloni (2000); Yalcin (2007,2015); Mandelkern (2019).

Under attitudes:

(4) a. #Suppose John is guilty but he might not be.
b. Suppose John is guilty but we don't know it.

Under disjunction:

(5) a. #Either John is guilty but might not be, or he's innocent but might not be.
b. Either John is guilty and we don't know it, or he's innocent and we don't know it.

Under quantifiers:

(6) a. #Everyone who is guilty might not be guilty.
b. Everyone who is guilty is, for all we know, not guilty.

Wittgenstein sentences

Order doesn't seem to matter for embeddings (*pace* dynamic treatments):

(7) a. #Suppose Sue might be sick but she isn't.
b. #Suppose Sue isn't sick but she might be.

(8) a. #It might be raining and it isn't, or it might be sunny and it isn't.
b. #It's not raining and it might be, or it's not sunny and it might be.

Wittgenstein sentences

It appears that Wittgenstein sentences are contradictions:[3] $p \wedge \Diamond\neg p \vdash \bot$.

[3] We use $p \wedge \Diamond\neg p$ as a stand-in for all Wittgenstein sentences.

Pseudocomplementation

…but they can't be in a classical setting, where negation is *pseudocomplementation*:

- $\neg$ is pseudocomplementation iff $\varphi \wedge \psi \vdash \bot$ entails $\psi \vdash \neg\varphi$;
- then from $p \wedge \Diamond\neg p \vdash \bot$ we would have $\Diamond\neg p \vdash \neg p$, absurdly.

Distributivity

For more evidence that epistemic modals introduce non-classicality, consider:

Distributivity: $\varphi \wedge (\psi \vee \chi) \dashv\vdash (\varphi \wedge \psi) \vee (\varphi \wedge \chi)$.

Distributivity appears to fail for epistemic modals.

(9) a. John might be guilty and might be innocent, and he is innocent or guilty.
b. #John is innocent and might be guilty, or he is guilty and might be innocent.

(9-b) is a disjunction of absurdities, but (9-a) entails (9-b) given distributivity.[4]

[4] Mandelkern 2019.

Distributivity

This failure of distributivity is plausibly related to puzzles involving quantification.[5] Consider a fair lottery where at least one ticket, but not all, won.

(10) Every ticket might be a losing ticket.

(11) #Some winning ticket might be a losing ticket.

Treat (10) as $\Diamond\neg W(t_1) \wedge \cdots \wedge \Diamond\neg W(t_n)$. We know $W(t_1) \vee \cdots \vee W(t_n)$. Hence

$$\big(\Diamond\neg W(t_1) \wedge \cdots \wedge \Diamond\neg W(t_n)\big) \wedge \big(W(t_1) \vee \cdots \vee W(t_n)\big).$$

Distributivity would allow us to infer

$$\big(W(t_1) \wedge \Diamond\neg W(t_1)\big) \vee \cdots \vee \big(W(t_n) \wedge \Diamond\neg W(t_n)\big), \text{ i.e., (11).}$$

[5] Ninan 2018, based on Aloni 2000.

Goal

Our goal: a theory of epistemic modals where

- WS's are *inconsistent* and *everywhere substitutable for contradictions*,
- negation is (therefore) not pseudocomplementation, and
- distributivity is not valid,

but which otherwise preserves (i) *all* of classical logic for the non-modal fragment;
(ii) all of classical logic for sentences at a given "epistemic level" (more on this later);
(iii) *as much as appears to still be valid* for sentences that cross epistemic levels.

Algebraic semantics

Ortholattices

Definition

An *ortholattice* is a tuple $\langle A, \vee, 0, \wedge, 1, \neg \rangle$ where $\langle A, \vee, 0, \wedge, 1 \rangle$ is a bounded lattice and $\neg$ is a unary operation on A, called the *orthocomplementation*, that satisfies:

1. complementation: for all $a \in A$, $a \vee \neg a = 1$ and $a \wedge \neg a = 0$;
2. involution: for all $a \in A$, $\neg\neg a = a$;
3. order-reversal: for all $a, b \in A$, if $a \leq b$, then $\neg b \leq \neg a$.

An equivalent definition replaces **??** with *De Morgan's laws*:

- for all $a, b \in A$, $\neg(a \vee b) = \neg a \wedge \neg b$;
- for all $a, b \in A$, $\neg(a \wedge b) = \neg a \vee \neg b$.

Ortholattices

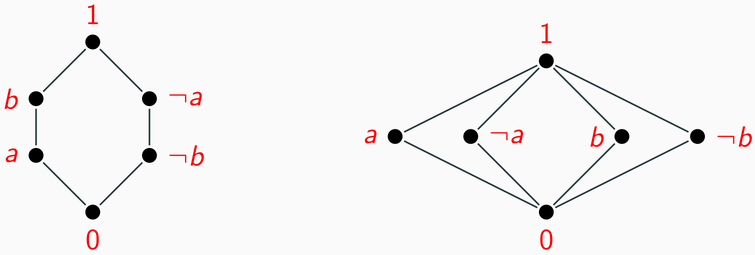


Figure 1: Hasse diagrams of the ortholattices $\mathbf{O}_6$ (left) and $\mathbf{MO}_2$ (right).

Ortholattics and Boolean algebras

Proposition

The following are equivalent:

1. L is a Boolean algebra.
2. L is an ortholattice that is distributive.
3. L is an ortholattice whose orthocomplementation $\neg$ is pseudocomplementation: $a \wedge b = 0$ implies $a \leq \neg b$.

Orthologic

Let $\mathcal{L}$ be the set of formulas generated by $\varphi ::= p \mid \neg\varphi \mid (\varphi \wedge \varphi)$ for $p \in \mathsf{Prop}$.

Definition (Goldblatt 1974)

An **orthologic** is a binary relation $\vdash$ on $\mathcal{L}$ such that for all $\varphi, \psi, \chi \in \mathcal{L}$:

1. $\varphi \vdash \top$;
2. $\varphi \vdash \varphi$;
3. $\varphi \wedge \psi \vdash \varphi$;
4. $\varphi \wedge \psi \vdash \psi$;
5. $\varphi \vdash \neg\neg\varphi$;
6. $\neg\neg\varphi \vdash \varphi$;
7. $\varphi \wedge \neg\varphi \vdash \psi$;
8. if $\varphi \vdash \psi$ and $\psi \vdash \chi$, then $\varphi \vdash \chi$;
9. if $\varphi \vdash \psi$ and $\varphi \vdash \chi$, then $\varphi \vdash \psi \wedge \chi$;
10. if $\varphi \vdash \psi$, then $\neg\psi \vdash \neg\varphi$.

As the intersection of orthologics is clearly an orthologic, there is a smallest orthologic, denoted O or $\vdash_{\mathsf{O}}$.

With $\varphi \vee \psi := \neg(\neg\varphi \wedge \neg\psi)$: $\varphi \vdash \varphi \vee \psi$; and if $\varphi \vdash \chi$ and $\psi \vdash \chi$, then $\varphi \vee \psi \vdash \chi$.

Epistemic ortholattices

Definition

A *modal ortholattice* is a tuple $\langle A, \vee, 0, \wedge, 1, \neg, \Box\rangle$ where $\langle A, \vee, 0, \wedge, 1, \neg\rangle$ is an ortholattice and $\Box$ is a unary operation on A satisfying:

- $\Box(a \wedge b) = \Box a \wedge \Box b$ for all $a, b \in A$;
- $\Box 1 = 1$.

For $a \in A$, we define $\Diamond a = \neg\Box\neg a$.

Definition

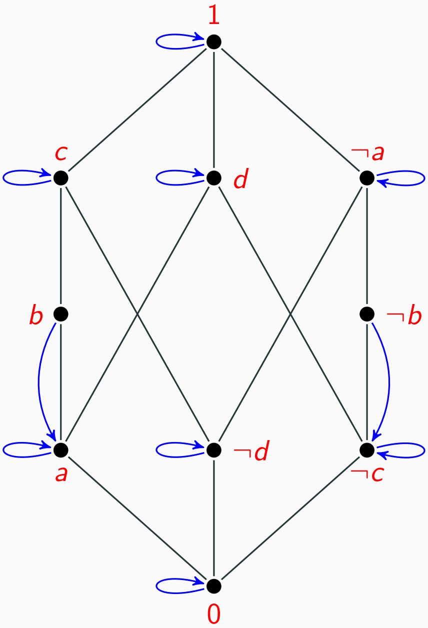
An *epistemic ortholattice* is a modal ortholattice also satisfying:

- T: $\Box a \leq a$ for all $a \in A$;
- Wittgenstein's Law: $\neg a \wedge \Diamond a = 0$ for all $a \in A$.

Epistemic ortholattices

Number of algebras of size *n* up to isomorphism:

	2	4	6	8	10
modal ortholattices	2	10	109	1,986	50,828
T modal ortholattices	1	3	21	221	3,285
epistemic ortholattices	1	1	4	23	207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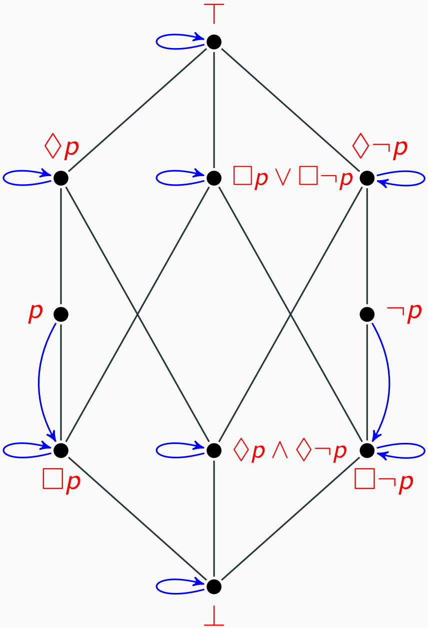
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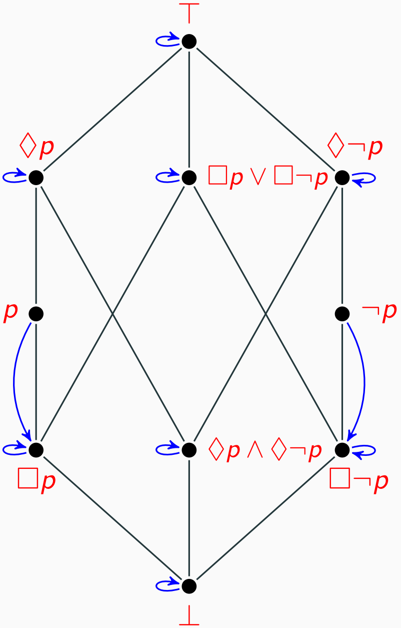
□¬p
⊥



Note the failure of distributivity:

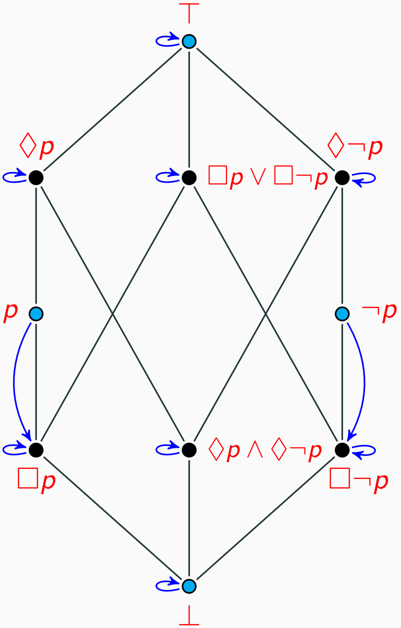
$(p \vee \neg p) \wedge (\Diamond p \wedge \Diamond \neg p) = \Diamond p \wedge \Diamond \neg p \neq 0$

and yet

$(p \wedge \Diamond \neg p) \vee (\neg p \wedge \Diamond p) = 0 \vee 0 = 0.$

Also note the failure of pseudo-complementation:

$p \wedge \Diamond \neg p = 0$ and yet $\Diamond \neg p \not\leq \neg p$.



Note the failure of distributivity:

$$(p \vee \neg p) \wedge (\Diamond p \wedge \Diamond\neg p) = \Diamond p \wedge \Diamond\neg p \neq 0$$

and yet

$$(p \wedge \Diamond\neg p) \vee (\neg p \wedge \Diamond p) = 0 \vee 0 = 0.$$

Also note the failure of pseudo-complementation:

$p \wedge \Diamond\neg p = 0$ and yet $\Diamond\neg p \not\leq \neg p$.

But non-epistemic propositions form a Boolean subalgebra.

Modal ortho-Boolean lattices

Definition

A *modal ortho-Boolean lattice* is a tuple $\langle A, B, \vee, 0, \wedge, 1, \neg, \Box\rangle$ where

- $\langle A, \vee, 0, \wedge, 1, \neg, \Box\rangle$ is a modal ortholattice and
- $\langle B, \vee_{|B}, 0, \wedge_{|B}, 1, \neg_{|B}\rangle$ is a Boolean algebra where $B \subseteq A$ and $\vee_{|B}$, $\wedge_{|B}$, and $\neg_{|B}$ are the restrictions of $\vee$, $\wedge$, and $\neg$, respectively, to B.

We interpret special Boolean propositional variables $\mathtt{p}, \mathtt{q}, \mathtt{r}, \dots$ in B, whereas arbitrary propositional variables $p, q, r, \dots$ can be interpreted as any elements of A.

A formula of the propositional modal language with $\Box$ (and $\Diamond\varphi := \neg\Box\neg\varphi$) is *Boolean* if all its propositional variables are Boolean and it does not contain $\Box$.

Level-wise Boolean

Definition

Given a modal ortho-Boolean lattice $L = \langle A, B, \vee, 0, \wedge, 1, \neg, \Box \rangle$, define:

- $B_0 = B$;
- B_{n+1} is the subortholattice of $\langle A, \vee, 0, \wedge, 1, \neg \rangle$ generated by $\{\Box b \mid b \in B_n\}$.

Then L is *level-wise Boolean* if each B_n is Boolean.

Motivation: no natural language counterexample to a classical inference that we have found is such that all propositions come from the same level B_n.

E.g., the counterexample to pseudocomplementation, going from $p \wedge \Diamond\neg p = 0$ to $\Diamond\neg p \leq \neg p$, involves $p, \neg p \in B_n$ and $\Diamond\neg p \in B_{n+1}$.

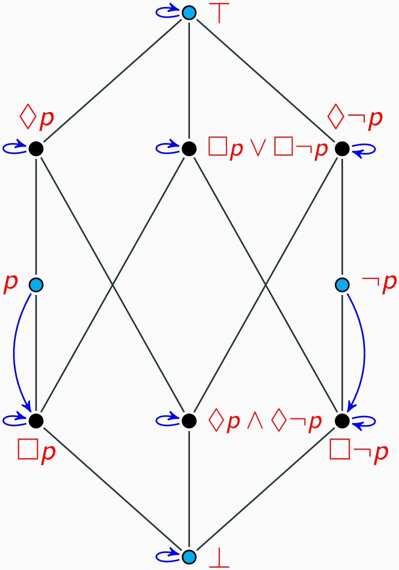
Picture that emerges: while classical reasoning across different epistemic levels is dangerous, classical reasoning within a given epistemic level is safe.

Epistemic ortho-Boolean lattice

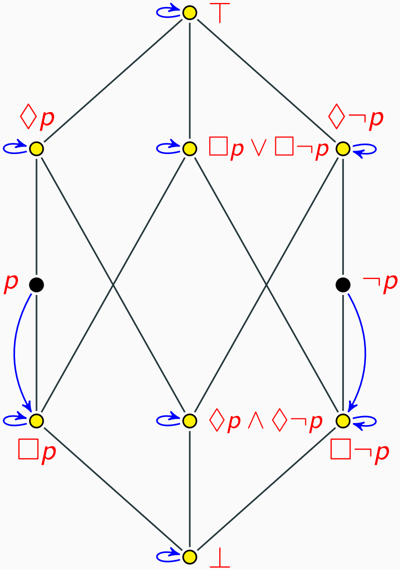
Definition

An *epistemic ortho-Boolean lattice* is a level-wise Boolean modal ortho-Boolean lattice $\langle A, B, \vee, 0, \wedge, 1, \neg, \Box \rangle$ in which $\langle A, \vee, 0, \wedge, 1, \neg, \Box \rangle$ is an epistemic ortholattice.

B_0 in cyan:



B_n for $n \geq 1$ in yellow:



Epistemic levels in language

Corresponding to the algebras B_n, we have a hierarchy of language fragments:

- Let $\mathcal{B}_0$ be the set of Boolean formulas.
- Let $\mathcal{B}_{n+1}$ be the smallest set of formulas that includes $\{\Box\varphi \mid \varphi \in \mathcal{B}_n\}$ and is closed under $\neg$ and $\wedge$.

The Epistemic Orthologic EO^+

1. $\varphi \vdash \top$;
2. $\varphi \vdash \varphi$;
3. $\varphi \wedge \psi \vdash \varphi$;
4. $\varphi \wedge \psi \vdash \psi$;
5. $\varphi \vdash \neg\neg\varphi$;
6. $\neg\neg\varphi \vdash \varphi$;
7. $\varphi \wedge \neg\varphi \vdash \psi$;
8. if $\varphi \vdash \psi$ and $\psi \vdash \chi$, then $\varphi \vdash \chi$;
9. if $\varphi \vdash \psi$ and $\varphi \vdash \chi$, then $\varphi \vdash \psi \wedge \chi$;
10. if $\varphi \vdash \psi$, then $\neg\psi \vdash \neg\varphi$.

11. if $\varphi \vdash \psi$, then $\Box\varphi \vdash \Box\psi$;
12. $\Box\varphi \wedge \Box\psi \vdash \Box(\varphi \wedge \psi)$;
13. $\varphi \vdash \Box\top$;
14. $\Box\varphi \vdash \varphi$;
15. $\neg\varphi \wedge \Diamond\varphi \vdash \bot$ (Wittgenstein's Law);

16. $\alpha \wedge (\beta \vee \gamma) \vdash (\alpha \wedge \beta) \vee (\alpha \wedge \gamma)$ for $\alpha, \beta, \gamma \in \mathcal{B}_n$.

Theorem

EO^+ *is the logic of epistemic ortho-Boolean lattices.*

Possibility semantics

Possibility semantics

Possibility semantics is a generalization of possible world semantics where possibilities are not assumed to satisfy the following property of possible worlds:

Primeness: a world makes a disjunction true only if it makes one of the disjuncts true.

More formally, possibility semantics starts with the following classic result:

Theorem

Let X be a nonempty set and c a closure operator on $\wp(X)$. Then the fixpoints of c, i.e., those $A \subseteq X$ with $c(A) = A$, ordered by $\subseteq$ form a complete lattice with

$$\bigwedge_{i \in I} A_i = \bigcap_{i \in I} A_i \text{ and } \bigvee_{i \in I} A_i = c\Big(\bigcup_{i \in I} A_i\Big).$$

A possibility semantics realizes the closure operator c in a concrete way, e.g., with a binary relation, and then adds further structure to interpret modalities.

Possibility semantics for orthologic

A *compatibility frame* is a pair $(S, \between)$ where $\between$ is a reflexive, symmetric relation on S.

Theorem (Birkhoff 1940, rephrased)

For any compatibility frame $(S, \between)$, the function $c : \wp(S) \to \wp(S)$ defined by

$$c_{\between}(A) = \{x \in S \mid \forall x' \between x \; \exists x'' \between x' : x'' \in A\}$$

is a closure operator on $\wp(S)$, whose fixpoints form a complete ortholattice $O(S, \between)$ with $\neg A = \{x \in S \mid \forall x' \between x \; x' \notin A\}$. We call the fixpoints $\between$-regular sets.

Thus, Birkhoff gives us a relational semantics for othologic: interpret propositional variables as $\between$-regular sets, $A \wedge B$ as $A \cap B$, $A \vee B$ as $c_{\between}(A \cup B)$, and $\neg A$ as above.

Theorem (MacLaren 1964)

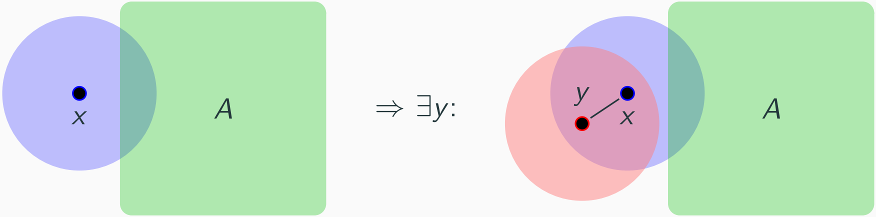
Let L be a complete ortholattice and V a join dense set of elements of L. Then L is isomorphic to $O(\mathrm{V} \setminus \{0\}, \between)$ where $a \between b$ iff $a \not\leq \neg b$.

Regularity

The intuition behind regularity is that if x does *not* make a proposition A true, then there should be a possibility y compatible with x that makes A *false*, so that all possibilities z compatible with y do not make A true. In a slogan:

Indeterminacy Implies Compatibility with Falsity.

Thus, if A is indeterminate at x, then x is compatible with a y that makes A false.



Possibility semantics for orthologic

Definition

A *compatibility model* is a pair $\mathcal{M} = \langle \mathcal{F}, V \rangle$ where $\mathcal{F} = \langle S, \between \rangle$ is a compatibility frame and V assigns to each $p \in \mathrm{Prop}$ a $\between$-regular set $V(p) \subseteq S$.

Definition

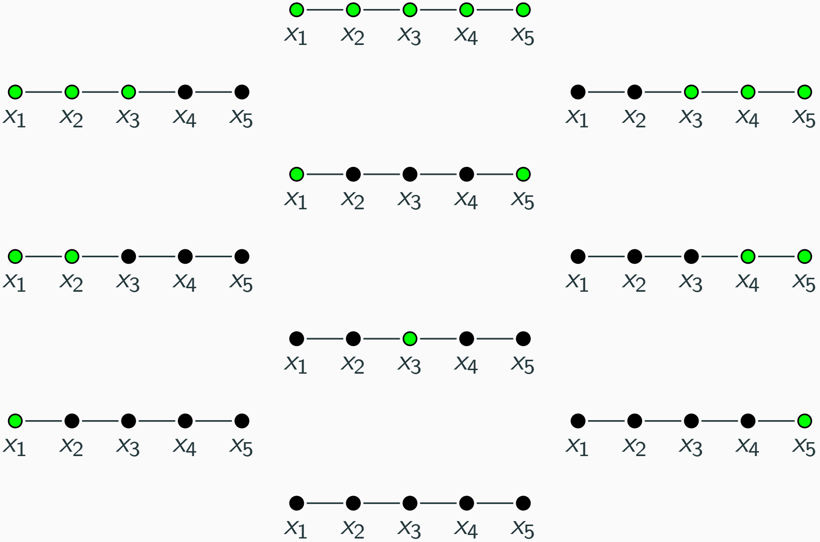
Given a model $\mathcal{M} = \langle S, \between, V \rangle$, $x \in S$, and $\varphi \in \mathcal{L}$, we define $\mathcal{M}, x \Vdash \varphi$ as follows:

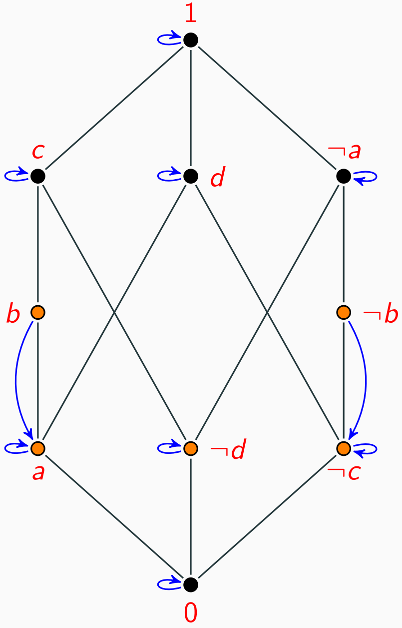
1. $\mathcal{M}, x \Vdash p$ iff $x \in V(p)$;
2. $\mathcal{M}, x \Vdash \neg\varphi$ for all $y \between x$, $\mathcal{M}, y \nVdash \varphi$;
3. $\mathcal{M}, x \Vdash \varphi \wedge \psi$ iff $\mathcal{M}, x \Vdash \varphi$ and $\mathcal{M}, x \Vdash \psi$;
4. $\mathcal{M}, x \Vdash \varphi \vee \psi$ iff $\forall x' \between x \; \exists x'' \between x'$: $\mathcal{M}, x'' \Vdash \varphi$ or $\mathcal{M}, x'' \Vdash \psi$.

Theorem (Goldblatt 1974)

The minimal orthologic O *is sound and complete with respect to the semantics above.*

Example

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Compatibility and refinement

Lemma

For any compatibility frame $\langle S, \between \rangle$, the following are equivalent for any $x, y \in S$:

1. *for all $\between$-regular sets $A \subseteq S$, if $x \in A$, then $y \in A$;*
2. *for all $z \in S$, if $z \between y$ then $z \between x$.*

*When these hold, we write $y \sqsubseteq x$ and say y **refines** x. Let $\downarrow x = \{y \in S \mid y \sqsubseteq x\}$.*

We say that two possibilities are **compossible** if they have a common refinement.

Lemma

For any compatibility frame $\langle S, \between \rangle$, the following are equivalent:

1. *$\mathcal{O}(S, \between)$ is a Boolean algebra;*
2. *any two possibilities that are compatible are compossible.*

Compatibility and refinement

For the compatibility frame

the refinement relation (with an arrow from y to z meaning $z \sqsubseteq y$) is

Possibility semantics for epistemic orthologic

To give possibility semantics for modal orthologic, we can add *accessibility relations* to our frames $(S, \between)$.

Definition

A **modal compatibility frame** is a triple $\mathcal{F} = \langle S, \between, R \rangle$ where $\langle S, \between \rangle$ is a compatibility frame and R is a binary relation on S satisfying the following condition, where $y \between_R x$ is an abbreviation for $\exists z$: $y \between z$ and $z \in R(x)$:

- R-regularity: if $y \between_R x$, then $\exists x' \between x \; \forall x'' \between x' \;\; y \between_R x''$
  (if A is a proposition, so is $\Box A$).

The frame is **epistemic** if R is reflexive and also satisfies

- Knowability: for all $x \in S$, there is a $y \in S$ such that $R(y) \subseteq {\downarrow}x$.
  (it is compatible with x that everything settled true by x is known).

Given a $\between$-regular set $A \subseteq S$, we define $\Box A = \{x \in S \mid R(x) \subseteq A\}$.

Possibility semantics for epistemic orthologic

Definition

A **modal compatibility frame** is a triple $\mathcal{F} = \langle S, \between, R\rangle$ where $\langle S, \between\rangle$ is a compatibility frame and R is a binary relation on S satisfying the following condition, where $y \between_R x$ is an abbreviation for $\exists z$: $y \between z$ and $z \in R(x)$:

- R-regularity: if $y \between_R x$, then $\exists x' \between x \; \forall x'' \between x' \;\; y \between_R x''$

The frame is **epistemic** if R is reflexive and also satisfies

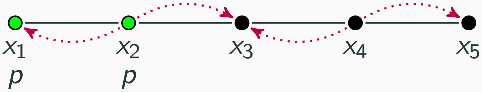
- Knowability: for all $x \in S$, there is a $y \in S$ such that for all $z \in R(y)$, $z \sqsubseteq x$.

Given a $\between$-regular set $A \subseteq S$, we define $\Box A = \{x \in S \mid R(x) \subseteq A\}$.

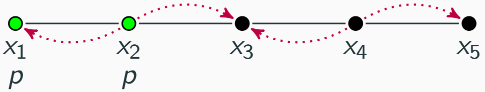
Proposition

For any such frame, $O(S, \between)$ equipped with $\Box$ operation is an epistemic ortholattice.

Example: The Epistemic Scale



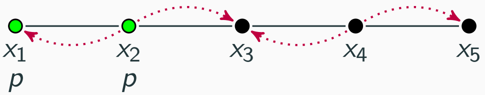
Example: The Epistemic Scale



Let's check Knowability: $\forall u \in S \ \exists v \colon R(v) \subseteq {\downarrow}u$.

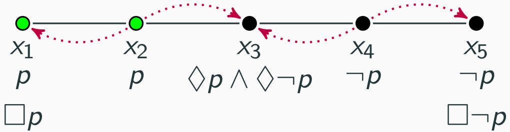
Indeed, take (x_2, x_1), (x_4, x_5), and (u, u) for $u \notin \{x_2, x_4\}$.

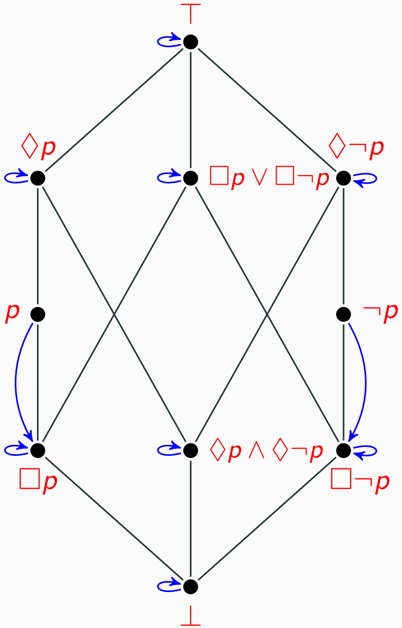
Example: The Epistemic Scale



- $[\![\Box p]\!]^{\mathcal{M}} = \{x_1\}$;
- $[\![\neg\Box p]\!]^{\mathcal{M}} = [\![\Diamond\neg p]\!]^{\mathcal{M}} = \{x_3, x_4, x_5\}$;
- $[\![\Box\neg p]\!]^{\mathcal{M}} = \{x_5\}$;
- $[\![\neg\Box\neg p]\!]^{\mathcal{M}} = [\![\Diamond p]\!]^{\mathcal{M}} = \{x_1, x_2, x_3\}$;
- $[\![\Diamond p \wedge \Diamond\neg p]\!]^{\mathcal{M}} = \{x_3\}$;
- $[\![\Box p \vee \Box\neg p]\!]^{\mathcal{M}} = \{x_1, x_5\}$.

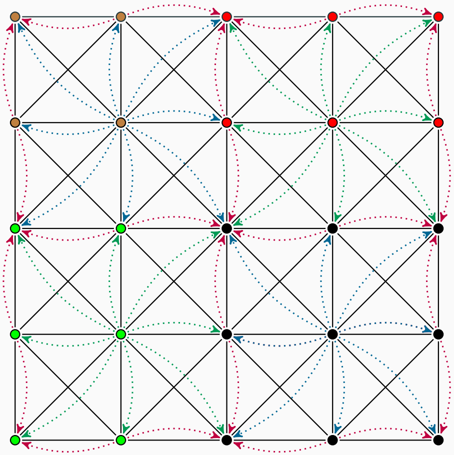
Example: The Epistemic Scale





$\top$
$\Diamond p$
$\Box p \vee \Box\neg p$
$\Diamond\neg p$
p
$\neg p$
$\Box p$
$\Diamond p \wedge \Diamond\neg p$
$\Box\neg p$
$\bot$

Example: The Epistemic Grid



Green possibilities make p true; red possibilities make q true; and brown possibilities makes both p and q true. The associated ortholattice has 1,942 elements.

Stratified frames

Definition

A *stratified epistemic compatibility frame* is a tuple $\mathcal{F} = \langle S, \between, R, \mathbb{B}\rangle$ where $\langle S, \between, R\rangle$ is an epistemic compatibility frame, $\mathbb{B}$ is a nonempty collection of $\between$-regular sets closed under $\cap$ and $\neg$, and where

- $\mathbb{B}_0 = \mathbb{B}$ and
- $\mathbb{B}_{n+1}$ is the closure of $\{\Box B \mid B \in \mathbb{B}_n\}$ under $\cap$ and $\neg$,

each $\mathbb{B}_n$ is such that for all $A, B \in \mathbb{B}_n$,

if there are $x \in A$ and $y \in B$ with $x \between y$, then $A \cap B \neq \varnothing$.

Proposition

In a stratified epistemic compatibility frame, each $\mathbb{B}_n$ forms a Boolean algebra under the operations $\cap$ and $\neg$.

Stratified frames

Definition

A *stratified epistemic compatibility frame* is a tuple $\mathcal{F} = \langle S, \between, R, \mathbb{B} \rangle$ where $\langle S, \between, R \rangle$ is an epistemic compatibility frame, $\mathbb{B}$ is a nonempty collection of $\between$-regular sets closed under $\cap$ and $\neg$, and where

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each $\mathbb{B}_n$ is such that for all $A, B \in \mathbb{B}_n$,

if there are $x \in A$ and $y \in B$ with $x \between y$, then $A \cap B \neq \varnothing$.

Models based on stratified frames interpret the Boolean propositional variables $\mathtt{p}, \mathtt{q}, \mathtt{r}, \ldots$ in $\mathbb{B}$.

Completeness

1. $\varphi \vdash \top$;
2. $\varphi \vdash \varphi$;
3. $\varphi \wedge \psi \vdash \varphi$;
4. $\varphi \wedge \psi \vdash \psi$;
5. $\varphi \vdash \neg\neg\varphi$;
6. $\neg\neg\varphi \vdash \varphi$;
7. $\varphi \wedge \neg\varphi \vdash \psi$;
8. if $\varphi \vdash \psi$ and $\psi \vdash \chi$, then $\varphi \vdash \chi$;
9. if $\varphi \vdash \psi$ and $\varphi \vdash \chi$, then $\varphi \vdash \psi \wedge \chi$;
10. if $\varphi \vdash \psi$, then $\neg\psi \vdash \neg\varphi$.

11. if $\varphi \vdash \psi$, then $\Box\varphi \vdash \Box\psi$;
12. $\Box\varphi \wedge \Box\psi \vdash \Box(\varphi \wedge \psi)$;
13. $\varphi \vdash \Box\top$;
14. $\Box\varphi \vdash \varphi$;
15. $\neg\varphi \wedge \Diamond\varphi \vdash \bot$ (Wittgenstein's Law);

16. $\alpha \wedge (\beta \vee \gamma) \vdash (\alpha \wedge \beta) \vee (\alpha \wedge \gamma)$ for $\alpha, \beta, \gamma \in \mathcal{B}_n$.

Theorem

EO^+ *is the logic of stratified epistemic compatibility frames.*

Constructing possibilities from worlds

Constructing possibilities from worlds

Starting with a set W of worlds, we will

construct possibilities as *pairs* (A, I) *of sets of worlds* where $\varnothing \neq A \subseteq I \subseteq W$.

In fact, our construction will apply starting with an arbitrary Boolean algebra B.

Definition

Let B be a Boolean algebra. The *epistemic frame of B* is the tuple $B^{\mathsf{e}} = (S, \between, R)$:

1. $S = \{(a, i) \mid a, i \in B, 0 \neq a \leq i\}$;
2. $(a, i) \between (a', i')$ iff $a \wedge a' \neq 0$ and $a \leq i'$ and $a' \leq i$;
3. $(a, i) R (a', i')$ iff $a \leq a'$ and $i' \leq i$.

Given a valuation $\theta : \mathsf{Bool} \to B$, we define θ^{e} by $\theta^{\mathsf{e}}(\mathrm{p}) = \{(a, i) \mid a \leq \theta(\mathrm{p})\}$.

Basic idea about a possibility (a, i):

- Boolean propositions that a entails *are true*;
- Boolean propositions consistent with a *might* be true;
- Boolean propositions that i entails *must* be true.

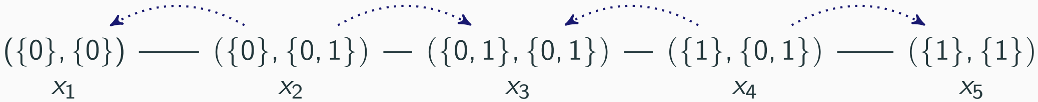
The clause for $\between$ ensures that if $\Diamond b$ is true at (a, i), then $\Box\neg b$ is not true at (a', i').

The clause for R ensures that $\Diamond b$ and $\Box b$ are preserved from (a, i) to (a', i').

Epistemic frame starting from two worlds

1. $S = \{(a, i) \mid a, i \in B, 0 \neq a \leq i\}$;
2. $(a, i) \between (a', i')$ iff $a \wedge a' \neq 0$ and $a \leq i'$ and $a' \leq i$;
3. $(a, i) R (a', i')$ iff $a \leq a'$ and $i' \leq i$.

Where $B = \wp(\{0, 1\})$, we have the following:



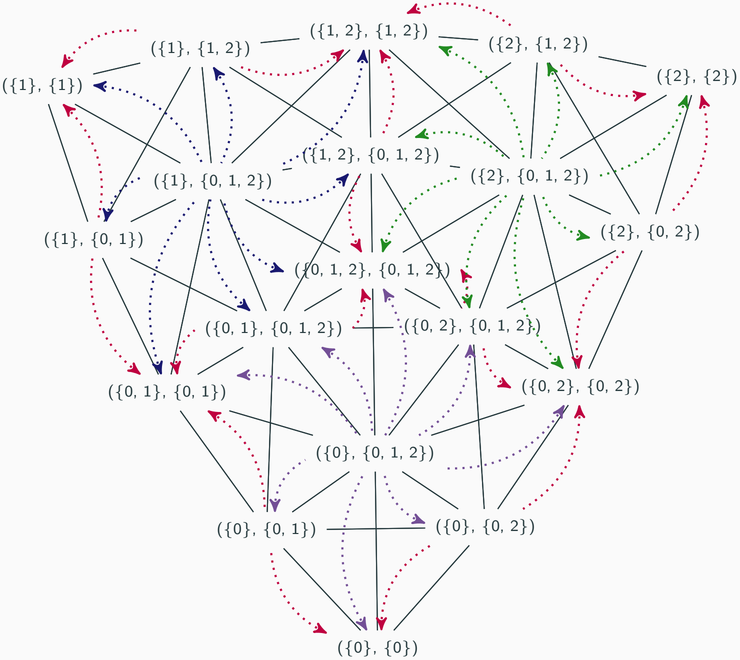
This is isomorphic to the Epistemic Scale!

Truth for Boolean formulas

Lemma

Let $M = (W, V)$ be a possible worlds model and $\mathcal{M}$ the epistemic model of $(\wp(W), V)$. For any Boolean formula φ and (A, I) in $\mathcal{M}$, we have:

1. $\mathcal{M}, (A, I) \Vdash \varphi$ iff for all $w \in A$, we have $M, w \vDash \varphi$;
2. $\mathcal{M}, (A, I) \Vdash \Box\varphi$ iff for all $w \in I$, we have $M, w \vDash \varphi$;
3. $\mathcal{M}, (A, I) \Vdash \Diamond\varphi$ iff for some $w \in A$, we have $M, w \vDash \varphi$.



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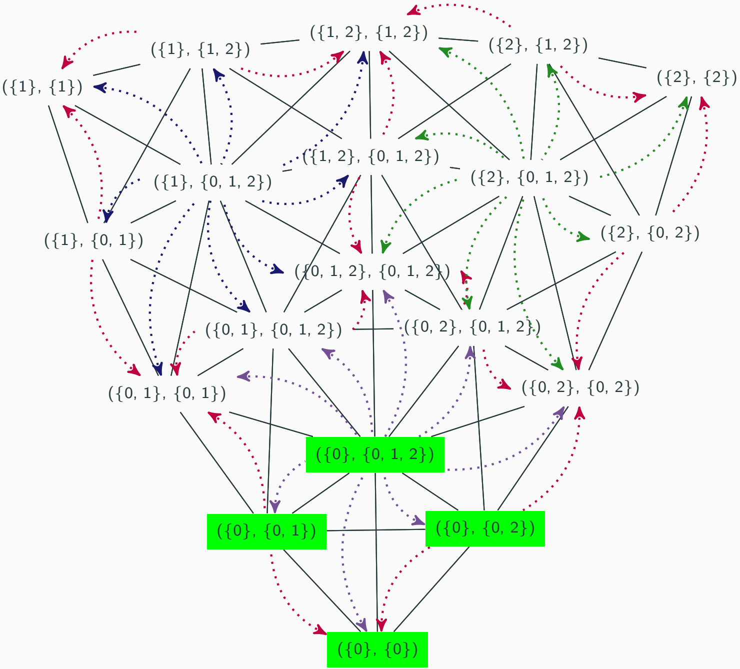
$[\![0]\!]^{\mathcal{M}}$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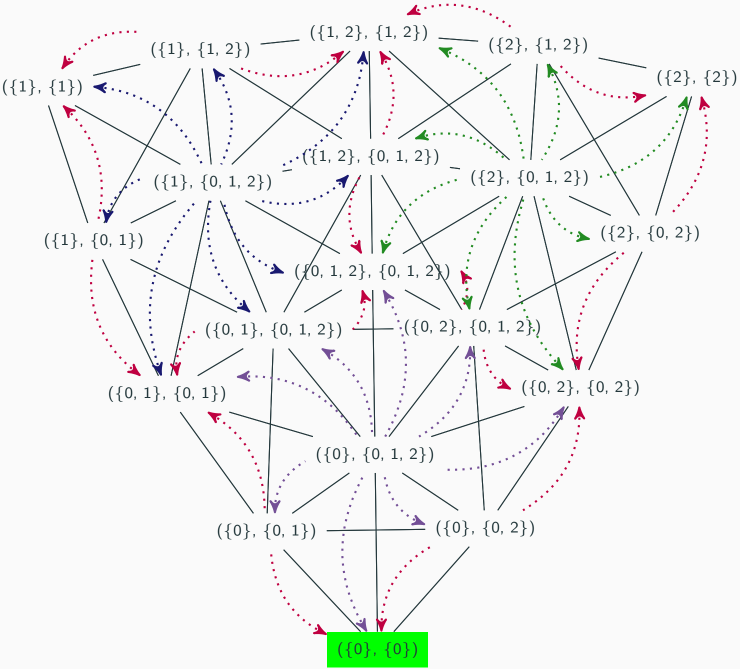
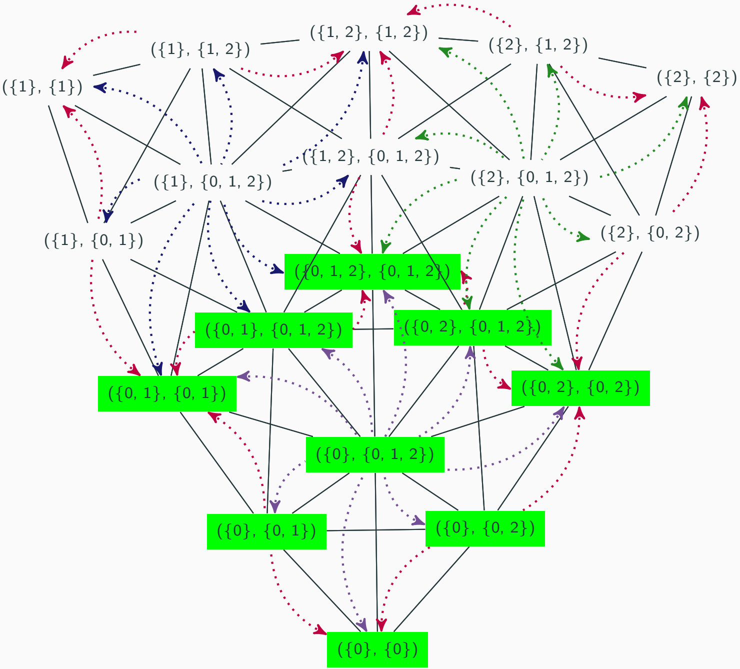
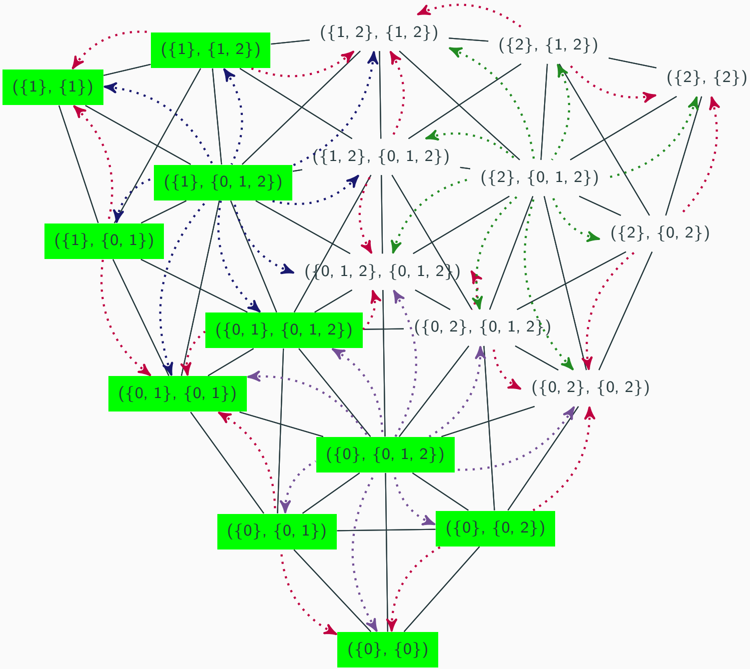
$[\![\Box 0]\!]^{\mathcal{M}}$



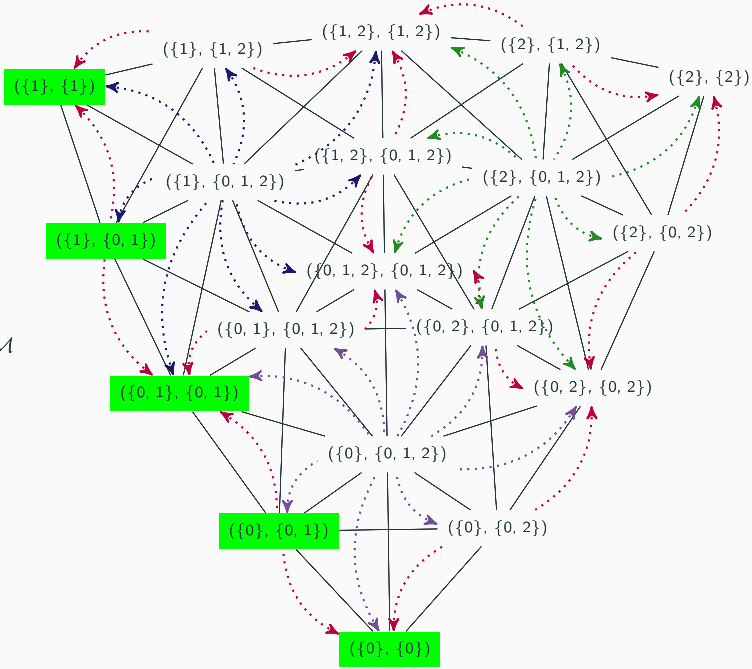
$[\![\Diamond 0]\!]^{\mathcal{M}}$



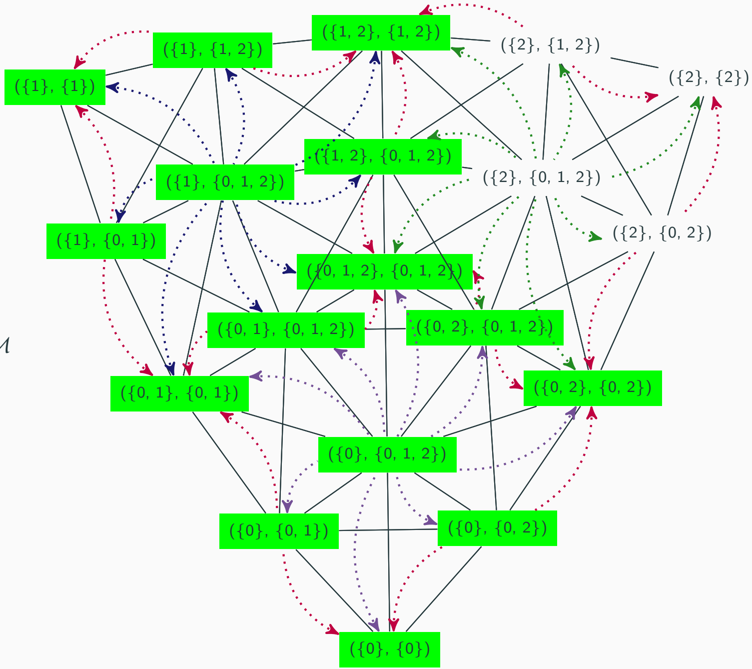
$[\![0 \vee 1]\!]^{\mathcal{M}}$

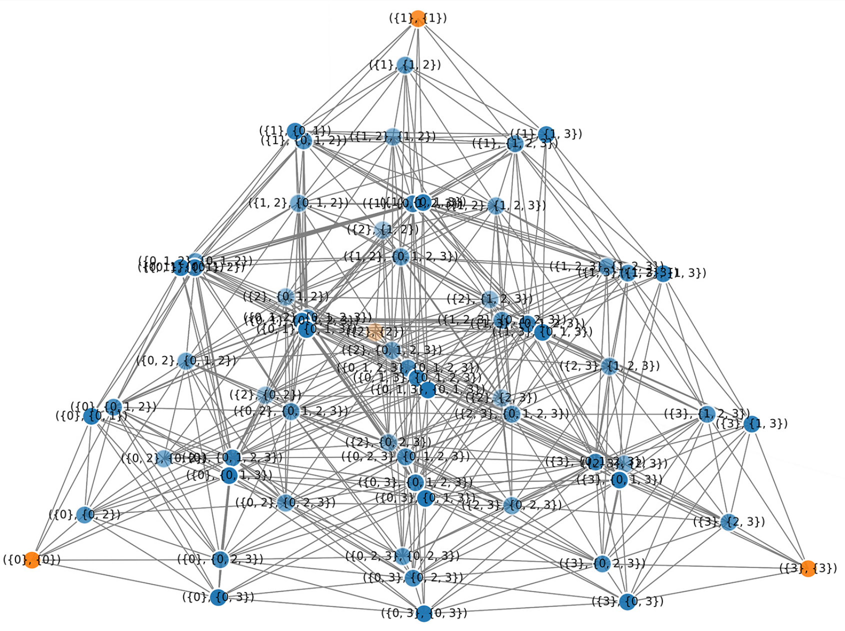


$[\![\Box(0 \vee 1)]\!]^{\mathcal{M}}$



$[\![\Diamond(0 \vee 1)]\!]^{\mathcal{M}}$

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Theorem

For any Boolean algebra B with lattice order $\leq$:

1. B^{e} is an epistemic compatibility frame;
2. the map e defined by $e_B(a) = \{(b, i) \in S \mid b \leq a\}$ is an embedding of B into the epistemic ortholattice $\mathcal{O}(B^{\mathsf{e}})$, which we therefore call the *epistemic extension of* B;
3. $\mathcal{O}(B^{\mathsf{e}})$ is an S5 epistemic ortholattice;
4. for all $b \in B$, if $b \notin \{0, 1\}$, then $\Diamond e_B(b) \not\leq e_B(b)$ in $\mathcal{O}(B^{\mathsf{e}})$.

With significant additional work, we prove the following.

Theorem

$(B^{\mathsf{e}}, \{e_B(b) \mid b \in B\})$ is a stratified epistemic compatibility frame.

Epistemic frames of BAs validate some additional laws for Boolean propositions, which arbitrary epistemic compatibility frames do not.

Proposition

For any Boolean algebra B and $U, U_1, U_2, V, V_1, V_2 \in \mathcal{O}(B^{\mathrm{e}})$ in the image of the embedding e_B:

1. $(U_1 \vee U_2) \wedge \lozenge(U_1 \wedge V_1) \wedge \lozenge(U_2 \wedge V_2) \subseteq (U_1 \wedge \lozenge V_1) \vee (U_2 \wedge \lozenge V_2)$;
2. $(U_1 \vee U_2) \wedge \Box V \subseteq (U_1 \wedge \Box V) \vee (U_2 \wedge \Box V)$;
3. $(U \wedge \lozenge V) \subseteq \lozenge(U \wedge V)$;
4. $(U \vee \lozenge V) \wedge \neg\lozenge V \subseteq U$;
5. $(U \vee \lozenge V) \wedge \neg U \subseteq \lozenge V$.

To do: prove the completeness of an extension of EO^{+} with respect to epistemic frames coming from Boolean algebras.

Conclusion

Further directions

In the paper, we also show

- how to lift *probability* from worlds to possibilities and
- how to lift *conditionals* from worlds to possibilities,

and we compare our approach to others.

For future work:

- *axiomatize* the logic of epistemic frames of Boolean algebras;
- study the interaction of *quantifiers* and modals/conditionals.

Appendix A: Lifting Probabilities

Lifting probabilities

Definition

Given a nonempty set W, distinguished information state $\mathcal{I} \subseteq W$, and a finitely additive probability measure $\mu : \wp(W) \to [0,1]$ with $\mu(\mathcal{I}) = 1$, we define the *epistemic extension* $\mu^{\mathsf{e}}_{\mathcal{I}} : O(\wp(W)^{\mathsf{e}}) \to [0,1]$ of μ with respect to U as follows:

- $\mu^{\mathsf{e}}_{\mathcal{I}}(U) = \mu\big(\bigcup\{A \subseteq W \mid (A, \mathcal{I}) \in U\}\big)$.

A natural choice of $\mathcal{I}$, at least in the finite case, is $\mathcal{I} = \{w \in W \mid \mu(\{w\}) > 0\}$.

Intuitively, to compute the probability of a proposition $U \in O(\wp(W)^{\mathsf{e}})$, we compute the probability of the worldly proposition obtained by unioning the first coordinates of those possibilities $(A, \mathcal{I}) \in U$. A useful fact is that this union is either empty or yields the *largest* A such that $(A, \mathcal{I}) \in X$.

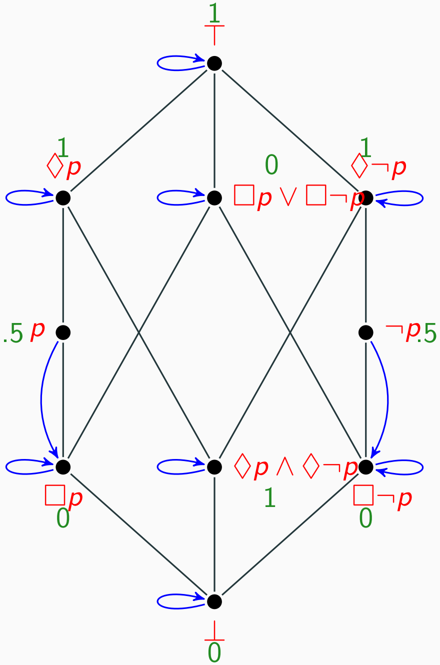
Example

Where $W = U = \{0, 1, 2\}$ and μ is the uniform measure with $\mu(\{0\}) = \mu(\{1\}) = \mu(\{2\}) = 1/3$, we obtain the probabilities in Table ??. E.g.,

$$\begin{aligned}
\mu^{\mathsf{e}}_W(\llbracket \Box 0 \rrbracket^{\mathcal{M}}) &= \mu\left(\bigcup \{A \subseteq W \mid (A, W) \in \llbracket \Box 0 \rrbracket^{\mathcal{M}}\}\right) \\
&= \mu\left(\bigcup \{A \subseteq W \mid (A, W) \in \{(\{0\}, \{0\})\}\}\right) \\
&= \mu(\varnothing) = 0.
\end{aligned}$$

formula φ	$\mu^{\mathsf{e}}_W(\llbracket \varphi \rrbracket^{\mathcal{M}})$
0	$1/3$
$\Box 0$	0
$\Diamond 0 \wedge \Diamond 1 \wedge \Diamond 2$	1
$0 \wedge \Diamond 1$	0

Table 1: Lifted probabilities given the uniform distribution on three worlds


1
⊤
1
$\Diamond p$
0
$\Box p \vee \Box \neg p$
1
$\Diamond \neg p$
.5 p
$\neg p$ .5
$\Diamond p \wedge \Diamond \neg p$
1
$\Box p$
0
$\Box \neg p$
0
⊥
0

Epistemic measures

Definition

A function μ from an epistemic ortho-Boolean lattice L to $[0,1]$ is an **epistemic measure** if for all $a, b \in L$,

(i) $a \leq b$ implies $\mu(a) \leq \mu(b)$,

(ii) $\mu(\neg a) = 1 - \mu(a)$,

(iii) $\mu(a) = 1$ and $\mu(b) = 1$ jointly imply $\mu(a \wedge b) = 1$, and

(iv) the restriction of μ to each $\mathbb{B}_n$ is a finitely additive probability measure.

Theorem

For W, $\mathcal{I}$, and μ as before, the lifted measure $\mu^{\mathsf{e}}_{\mathcal{I}}$ agrees with μ on $\mathbb{B}_0$, satisfies (i), (ii), and (iv), and it satisfies (iii) iff $\mathcal{I} = \{w \in W \mid \mu(\{w\}) > 0\}$.

Appendix B: Lifting Conditionals

Lifting conditionals

Recall that given a set-selection function $h : (X \times \mathcal{P}) \to \wp(X)$, we define a conditional operation on the set $\mathcal{P}$ of propositions by

$$U \to_h V = \{x \in X \mid f(x, U) \subseteq V\}.$$

Definition

Let W be a nonempty set, $f : (W \times \wp(W)) \to \wp(W)$ a set-selection function, and S the set of possibilities in the epistemic extension $\wp(W)^{\mathsf{e}}$. Then a set-selection function $g : (S \times O(\wp(W)^{\mathsf{e}})) \to \wp(S)$ is *an epistemic extension of f* if for all nonempty $C \subseteq W$, we have

$$g((A, I), e(C)) = \{(\bigcup\{f(w, C) \mid w \in A\}, \bigcup\{f(w, C) \mid w \in I\})\}$$

where e is the embedding from the epistemic extension theorem.

Definition

Let W be a nonempty set, $f : (W \times \wp(W)) \to \wp(W)$ a set-selection function, and S the set of possibilities in the epistemic extension $\wp(W)^{\mathrm{e}}$. Then a set-selection function $g : (S \times O(\wp(W)^{\mathrm{e}})) \to \wp(S)$ is *an epistemic extension of f* if for all nonempty $C \subseteq W$, we have

$$g((A, I), e(C)) = \left\{ \left(\bigcup\{f(w, C) \mid w \in A\}, \bigcup\{f(w, C) \mid w \in I\} \right) \right\}$$

where e is the embedding from the epistemic extension theorem.

Proposition

If W, f, and g are as above, then the embedding e from the epistemic extension theorem also preserves the conditional, i.e., for all $C, D \in \wp(W)$:

$$e(C \to_f D) = e(C) \to_g e(D).$$

Scopelessness

A desirable prediction of this approach to modals and conditionals is the following scopelessness property, which implies that for non-modal φ and ψ,

$$\Box(\varphi \to \psi) \text{ is equivalent to } \varphi \to \Box\psi.$$

Proposition

If W, f, g, and e are as in the previous proposition, then for all $C, D \in \wp(W)$,

$$\Box(e(C) \to_g e(D)) = e(C) \to_g \Box e(D).$$

From sequences of worlds to possibilities

Definition

Given a countable set W of worlds, let W^* be the set of all sequences (indexed by an initial segment of $\mathbb{N}$) that list all elements of W without repetition. Given a proposition $\mathcal{U} \subseteq W^*$, let

$$\mathcal{U}_{\downarrow} = \{w \in W \mid \text{some sequence in } \mathcal{U} \text{ starts with } w\}.$$

Define a set-selection function $f : (W^* \times \wp(W^*)) \to W^*$ as follows:

1. $f(s, \mathcal{A}) = \varnothing$ if $\mathcal{A} = \varnothing$;
2. otherwise $f(s, \mathcal{A})$ is the singleton set of the sequence obtained from s by putting all worlds in $\mathcal{A}_{\downarrow}$, ordered as in s, before all worlds not in $\mathcal{A}_{\downarrow}$, ordered as in s.

From sequences of worlds to possibilities

Definition

For finite W, given a probability measure μ on $\wp(W)$, let μ^* be the measure on $\wp(W^*)$ such that the probability of a sequence $s \in W^*$ is the probability of obtaining s by sampling without replacement from W according to μ.

From sequences of worlds to possibilities

Now we do the following:

1. Construct the epistemic frame $\wp(W^*)^{\mathsf{e}}$;
2. Construct the lifted epistemic measure $(\mu^*)^{\mathsf{e}}$ on $O(\wp(W^*)^{\mathsf{e}})$;
3. Construct the minimal epistemic extension g of the set-selection function f we defined on W^*.

So the picture is this:

worlds	$\rightarrow$	sequences	$\rightarrow$	possibilities
W	$\rightarrow$	W^*	$\rightarrow$	$\wp(W^*)^{\mathsf{e}}$
μ	$\rightarrow$	μ^*	$\rightarrow$	$(\mu^*)^{\mathsf{e}}$
		f	$\rightarrow$	g

Results for three worlds

- Probability of $[\![0]\!]^{\mathcal{M}}$ is 1/3;
- Probability of $[\![(0 \vee 1) \to 0]\!]^{\mathcal{M}}$ is 1/2;
- Probability of $[\![\Diamond((0 \vee 1) \to 0)]\!]^{\mathcal{M}}$ is 1;
- Probability of $[\![\Box((0 \vee 1) \to 0)]\!]^{\mathcal{M}}$ is 0;
- Probability of $[\![(0 \vee 1) \to \Box 0]\!]^{\mathcal{M}}$ is 0 (equivalent to $\Box((0 \vee 1) \to 0)$);
- Probability of $[\![\neg(1 \vee 2) \to \Box 0]\!]^{\mathcal{M}}$ is 1 (true at all possibilities);
- Probability of $[\![0 \to ((0 \vee 1) \to 0))]\!]^{\mathcal{M}}$ is 1 (true at all possibilities);
- Probability of $[\![((0 \vee 1 \vee 2) \to 0) \to 0]\!]^{\mathcal{M}}$ is 1 (true at all possibilities);
- Probability of $[\![0 \to \Diamond \neg 0]\!]^{\mathcal{M}}$ is 0 (true at no possibilities).

Modal antecedents

So far we have not said how to handle modal antecedents.

Given a proposition $U \in O(\wp(W)^{\mathsf{e}})$, we define its *worldly projection* as

$$U_{\Downarrow} = \bigcup\{A \subseteq W \mid \exists I : (A, I) \in U\}.$$

Definition

Given W, f, and S (the set of possibilities in the epistemic frame of $\wp(W)$) as before and $d : (S \times S) \to \mathbb{R}_{\geq 0}$, we define a set-selection function $f^d : (S \times O(\wp(W)^{\mathsf{e}})) \to \wp(S)$ by

$$f^d((A, I), U) = \operatorname*{arg\,min}_{(A', I') \in \Box U} d\Big((A', I'), \left(\bigcup\left\{f(w, U_{\Downarrow}) \mid w \in A\right\}, \bigcup\{f(w, U_{\Downarrow}) \mid w \in I\}\right)\Big).$$

Example

Recall that the *Hamming distance* between two sets X and Y, $d_H(X, Y)$, is the cardinality of the symmetric difference $(X \setminus Y) \cup (Y \setminus X)$. We lift this to a distance between possibilities by summing pointwise Hamming distances:

$$d_H((A, I), (A', I')) = d_H(A, A') + d_H(I, I').$$

Table ?? gives examples of Hamming distances between possibilities in the epistemic frame constructed from two worlds.

(A, I)	(A', I')	$d_H((A, I), (A', I'))$
$(\{0\}, \{0, 1\})$	$(\{0, 1\}, \{0, 1\})$	1
$(\{0\}, \{0, 1\})$	$(\{0\}, \{0\})$	1
$(\{0\}, \{0, 1\})$	$(\{1\}, \{0, 1\})$	2
$(\{0\}, \{0, 1\})$	$(\{1\}, \{1\})$	3

Table 2: Hamming distances between possibilities in the epistemic frame from two worlds.

Results for three worlds

- Probability of $[\![(\Box 0 \vee \Box 1) \rightarrow \Box 0]\!]^{\mathcal{M}}$ is 1/2;
- Probability of $[\![\Box(0 \vee 1) \rightarrow \Box 0]\!]^{\mathcal{M}}$ is 0;
- Probability of $[\![(1 \vee 2)]\!]^{\mathcal{M}}$ is 2/3;
- Probability of $[\![\Diamond\neg 0 \rightarrow (1 \vee 2)]\!]^{\mathcal{M}}$ is 2/3;
- Probability of $[\![\Diamond 0 \rightarrow \neg 0]\!]^{\mathcal{M}}$ is 0 (true at no possibilities);
- Probability of $[\![\Diamond 0 \rightarrow 0]\!]^{\mathcal{M}}$ is 1/3;
- Probability of $[\![(0 \wedge \Diamond\neg 0) \rightarrow \bot]\!]^{\mathcal{M}}$ is 1 (true at all possibilities).

Appendix C: Natural deduction

A Fitch-style natural deduction system for orthologic can be obtained from one for classical logic by dropping Fitch's rule of Reiteration, which we can see is unacceptable for a language with epistemic modals:

1	$\Diamond p \wedge (p \vee \neg p)$	
2	$\Diamond p$	$\wedge$E, 1
3	$(p \vee \neg p)$	$\wedge$E, 1
4	$\quad p$	
5	$\quad p \vee (\neg p \wedge \Diamond p)$	$\vee$I, 4
6	$\quad \neg p$	
7	$\quad \Diamond p$	Reiteration, 2
8	$\quad \neg p \wedge \Diamond p$	$\wedge$I, 6, 7
9	$\quad p \vee (\neg p \wedge \Diamond p)$	$\vee$I, 8
10	$p \vee (\neg p \wedge \Diamond p)$	$\vee$E, 3, 4–5, 6–9

A Fitch-style natural deduction system for orthologic can be obtained from one for classical logic by dropping Fitch's rule of Reiteration, which we can see is unacceptable for a language with epistemic modals.

Then one can add Fitch's (1966) Intro and Elim rules for $\Box$, plus the following:

$$
\begin{array}{l|ll}
\vdots & \vdots & \\
i & \neg\varphi \;\; (\Diamond\varphi) & \\
\vdots & \vdots & \\
j & \Diamond\varphi \;\; (\neg\varphi) & \\
\vdots & \vdots & \\
k & \psi & \text{WL}, i, j
\end{array}
$$

A fundamental non-classical logic

If, following the intuitionists, we also drop Reductio Ad Absurdum from the Fitch-style natural deduction system for orthologic, then we obtain a logic based solely on the introduction and elimination rules for the connectives.

The paper "A fundamental non-classical logic" defines this logic and gives semantics for it (and variants) using a compatibility relation $\lhd$ that is not necessarily symmetric.