Quantification in Possibility Semantics

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ESSLLI 2023

Outline

Propositional quantification

Necessity and Possibility

Let's start in the basic propositional unimodal language.

Let's define what it is for a possibility x in a frame $\mathcal{F} = (S, \sqsubseteq)$ to settle a formula as true under a valuation v for $\mathcal{F}$ as follows:

$\dots$

- $\mathcal{F}, x \Vdash_v \Box\varphi$ iff for all $y \in S$: $\mathcal{F}, y \Vdash_v \varphi$;
- $\mathcal{F}, x \Vdash_v \Diamond\varphi$ iff for some $y \in S$: $\mathcal{F}, y \Vdash_v \varphi$.

Theorem. The set of valid formulas is axiomatized by the classical modal logic **S5**.

Propositional Quantifiers

Let's now add propositional quantifiers ($\forall p$, $\exists p$) to our language.

We define what it is for a possibility x in a frame $\mathcal{F} = (S, \sqsubseteq)$ to settle a formula as true under a valuation v for $\mathcal{F}$ as follows:

...

- $\mathcal{F}, x \Vdash_v \forall p\varphi$ iff for all valuations $u \sim_p v$: $\mathcal{F}, x \Vdash_u \varphi$;
- $\mathcal{F}, x \Vdash_v \exists p\varphi$ iff $\forall x' \sqsubseteq x\ \exists x'' \sqsubseteq x'\ \exists u \sim_p v$: $\mathcal{F}, x'' \Vdash_u \varphi$.

Propositional Quantifiers

Theorem (Holliday 2017)

The set of formulas valid according to the above semantics is axiomatized by the logic $\mathbf{S5\Pi}$ of Bull and Fine, which adds to **S5** the following axioms and rule for the propositional quantifiers:

- $\forall$-distribution: $\forall p(\varphi \to \psi) \to (\forall p\varphi \to \forall p\psi)$.
- $\forall$-instantiation: $\forall p\varphi \to \varphi^p_\psi$ where ψ is free for p in φ;
- Vacuous-$\forall$: $\varphi \to \forall p\varphi$ where p is not free in φ.
- $\forall$-generalization: if φ is a theorem, so is $\forall p\varphi$.

By contrast, if we restrict to possible world frames one obtains additional validities not derivable in $\mathbf{S5\Pi}$, such as:

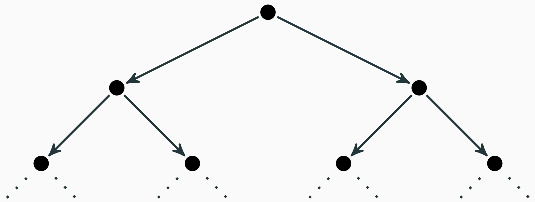
$$\exists q\big(q \wedge \forall p(\Box(q \to p) \vee \Box(q \to \neg p))\big).$$

Example

In the full infinite binary tree, no possibility satisfies

$$\exists q \big(q \wedge \forall p (\Box (q \to p) \vee \Box (q \to \neg p))\big),$$

since every proposition can be strengthened to a smaller one.



Where Q is any regular set, take any $x \in Q$ and $y \sqsubset x$.

The set $\downarrow y = \{z \in S \mid z \sqsubseteq y\}$ is regular, and $\downarrow y \subsetneq Q$.

First-order quantification

From Boolean algebras to first-order logic

Let us now turn to possibility semantics for FOL.

Versions were developed by Johan van Benthem in 1981 (see his "Tales from an Old Manuscript") and Matthew Harrison-Trainor in 2016 (see his "First-order possibility models and finitary completeness proofs").

The version I will present is based on a section of my chapter on "Possibility Semantics" for *Research Trends in Contemporary Logic*, eds. Fitting et al.

Choice-free model theory?

The traditional completeness theorem for first-order logic for uncountable languages (Malcev 1936), stating that

every consistent set of first-order sentences has a *Tarskian model*,

is not provable in ZF, as it is equivalent in ZF to the Boolean Prime Filter Theorem (Henkin 1954) (for a proof, see, e.g., Bell and Slomson 1974, p. 104).

By contrast, we will prove **in ZF** that for arbitrary languages,

every consistent set of FO-sentences has a *possibility model*.

Definition

A *first-order possibility model for L* is a tuple $\mathfrak{A} = (S, \leqslant, D, \asymp, I)$:

1. $(S, \leqslant)$ is a poset, and D is a nonempty set;
2. $\asymp$ assigns to $s \in S$ an equivalence relation $\asymp_s$ on D s.th.:
   - *persistence* for $\asymp$: if $a \asymp_s b$ and $s' \geqslant s$, then $a \asymp_{s'} b$;
   - *refinability* for $\asymp$: if $a \not\asymp_s b$, then $\exists s' \geqslant s \; \forall s'' \geqslant s' \; a \not\asymp_{s''} b$.
3. I assigns to each pair of an n-ary predicate R of L and $s \in S$ a set $I(R, s) \subseteq D^n$ and to each n-ary function symbol f of L and $s \in S$ a set $I(f, s) \subseteq D^{n+1}$ s. th.:
   - *persistence* for R: if $\overline{a} \in I(R, s)$, $s' \geqslant s$, and $\overline{a} \asymp_{s'} \overline{b}$, then $\overline{b} \in I(R, s')$;
   - *refinability* for R: if $\overline{a} \notin I(R, s)$, then $\exists s' \geqslant s \; \forall s'' \geqslant s' \; \overline{a} \notin I(R, s'')$;
   - *persistence* for f: if $\overline{a} \in I(f, s)$, $s' \geqslant s$, and $\overline{a} \asymp_{s'} \overline{b}$, then $\overline{b} \in I(\sigma, s')$;
   - *quasi-functionality* for f: if $(\overline{a}, b), (\overline{a}, b') \in I(f, s)$, then $b \asymp_s b'$;
   - *definedness* for f: $\forall \overline{a} \in D^n \; \exists s' \geqslant s, b \in D$: $(\overline{a}, b) \in I(f, s)$.

Definition

A model is *everywhere defined* (ED) if for each n-ary function symbol f of L, $s \in S$, and $\overline{a} \in D^n$, there is $b \in D$: $(\overline{a}, b) \in I(f, s)$.

A *Tarskian model* is an ED first-order possibility model in which S contains only one possibility s, and $\asymp_s$ is the identity relation.

For simplicity, in the rest of this talk I focus on ED models.

Definition

A *pointed model* is a pair $\mathfrak{A}, s$ of a possibility model $\mathfrak{A}$ and possibility s in $\mathfrak{A}$.

Definition

Given a first-order possibility model $\mathfrak{A} = (S, \leqslant, D, \asymp, I)$, $s \in S$, and variable assignment $g : \mathrm{Var}(L) \to D$, we define a function $[\![\;]\!]_{\mathfrak{A},s,g} : \mathrm{Term}(L) \to \wp(D)$ recursively as follows:

1. $[\![x]\!]_{\mathfrak{A},s,g} = \{a \in D \mid a \asymp_s g(x)\}$ for $x \in \mathrm{Var}(L)$;
2. for an n-ary function symbol f and $t_1, \ldots, t_n \in \mathrm{Term}(L)$,

$$[\![f(t_1, \ldots, t_n)]\!]_{\mathfrak{A},s,g} = \{b \in D \mid \exists a_1, \ldots, a_n : a_i \in [\![t_i]\!]_{\mathfrak{A},s,g} \text{ and } (a_1, \ldots, a_n, b) \in I(f, s)\}.$$

Lemma

For any ED model $\mathfrak{A} = (S, \leqslant, D, \asymp, I)$, $s \in S$, variable assignment $g : \mathrm{Var}(L) \to D$, $t \in \mathrm{Term}(L)$, and $x \in \mathrm{Var}(L)$:

1. $[\![t]\!]_{\mathfrak{A},s,g}$ is an $\asymp_s$-equivalence class;
2. if $a \in [\![t]\!]_{\mathfrak{A},s,g}$ and $s' \sqsubseteq s$, then $a \in [\![t]\!]_{\mathfrak{A},s',g}$.

Definition

Given $\mathfrak{A} = (S, \leqslant, D, \asymp, I)$, define a function $I_{\asymp}$ that assigns to each pair of an n-ary predicate R of L and $s \in S$ a set $I_{\asymp}(R, s) \subseteq (D/\asymp_s)^n$ by

$$(\xi_1, \ldots, \xi_n) \in I_{\asymp}(R, s) \text{ iff } \exists a_1, \ldots, a_n : a_i \in \xi_i \text{ and } (a_1, \ldots, a_n) \in I(R, s).$$

Definition

Given an ED model $\mathfrak{A} = (S, \leqslant, D, \asymp, I)$ for L, formula φ of L, $s \in S$, and variable assignment $g : \mathrm{Var}(L) \to D$, we define the satisfaction relation $\mathfrak{A}, s \vDash_g \varphi$:

1. $\mathfrak{A}, s \vDash_g t_1 = t_2$ iff $[\![t_1]\!]_{\mathfrak{A},s,g} = [\![t_2]\!]_{\mathfrak{A},s,g}$;
2. $\mathfrak{A}, s \vDash_g R(t_1, \ldots, t_n)$ iff $([\![t_1]\!]_{\mathfrak{A},s,g}, \ldots, [\![t_n]\!]_{\mathfrak{A},s,g}) \in I_{\asymp}(R, s)$;
3. $\mathfrak{A}, s \vDash_g \neg\varphi$ iff for all $s' \geqslant s$, $\mathfrak{A}, s' \nvDash_g \varphi$;
4. $\mathfrak{A}, s \vDash_g \varphi \wedge \psi$ iff $\mathfrak{A}, s \vDash_g \varphi$ and $\mathfrak{A}, s \vDash_g \psi$;
5. $\mathfrak{A}, s \vDash_g \forall x \varphi$ iff for all $a \in D$, $\mathfrak{A}, s \vDash_{g[x:=a]} \varphi$.

A set Γ of formulas is *satisfiable in* $\mathfrak{A}$ if there is some possibility s in $\mathfrak{A}$ and variable assignment g such that $\mathfrak{A}, s \vDash_g \varphi$ for all $\varphi \in \Gamma$

Lemma

For any model $\mathfrak{A} = (S, \leqslant, D, \asymp, I)$ for L, variable assignment $g : \mathrm{Var} \to D$, and formulas φ, ψ of L:

$$\begin{aligned} \|\varphi\|_{\mathfrak{A},g} &:= \{s \in S \mid \mathfrak{A}, s \vDash_g \varphi\} \in \mathcal{RO}(S, \leqslant) \\ \|\neg\varphi\|_{\mathfrak{A},g} &= \neg\|\varphi\|_{\mathfrak{A},g} \\ \|\varphi \wedge \psi\|_{\mathfrak{A},g} &= \|\varphi\|_{\mathfrak{A},g} \wedge \|\varphi\|_{\mathfrak{A},g} \\ \|\forall x\varphi\|_{\mathfrak{A},g} &= \bigwedge\{\|\varphi\|_{\mathfrak{A},g[x:=a]} \mid a \in D\}. \end{aligned}$$

Theorem (Soundness)

For any set Γ of formulas and formula φ, if $\Gamma \vdash \varphi$, then for every pointed model $\mathfrak{A}, s$ and variable assignment g, if $\mathfrak{A}, s \vDash_g \psi$ for all $\psi \in \Gamma$, then $\mathfrak{A}, s \vDash_g \varphi$.

Henkinization

Definition

Given any first-order language L, we define a countable sequence of languages by:

$$\begin{aligned} L_0 &= L \\ L_{n+1} &= \text{extension of } L_n \text{ with new constant } c_{\exists x\varphi} \\ & \text{for each sentence } \exists x\varphi \text{ of } L_n \\ L_\omega &= \bigcup_{n\in\omega} L_n. \end{aligned}$$

Lemma

(ZF) For every consistent L-theory Γ, the set

$$H(\Gamma) = \mathrm{Cn}(\Gamma \cup \{\exists x\varphi \to \varphi^x_{c_{\exists x\varphi}} \mid \exists x\varphi \text{ a sentence of } L_\omega\})$$

is a consistent Henkinized L_ω-theory.

Canonical model for L

Definition

The *canonical model for* L is the tuple $\mathfrak{A}_L = (S, \leqslant, D, \asymp, I)$ where:

1. S is the set of all consistent L-theories;
2. $\Gamma \leqslant \Gamma'$ iff $\Gamma \subseteq \Gamma'$;
3. D is the set of closed terms of L_ω;
4. $t \asymp_\Gamma t'$ iff $t = t' \in H(\Gamma)$;
5. for any n-ary predicate symbol R and $\Gamma \in S$,

$$I(R, \Gamma) = \{(t_1, \dots, t_n) \mid R(t_1, \dots t_n) \in H(\Gamma)\};$$

6. for any n-ary function symbol f and $\Gamma \in S$,

$$I(f, \Gamma) = \{(t_1, \dots, t_{n+1}) \mid f(t_1, \dots, t_n) = t_{n+1} \in H(\Gamma)\}.$$

Choice-free completeness

Lemma

The canonical model for L is an ED first-order possibility model.

Lemma (Truth Lemma)

For every sentence φ of L and $\Gamma \in S$,

$$\mathfrak{A}_L, \Gamma \vDash \varphi \text{ iff } \varphi \in H(\Gamma).$$

Theorem (Strong Completeness)

(**ZF**) Every consistent L-theory Γ is satisfiable in the canonical first-order possibility model for L.

Choice-free model theory?

Hodges (*Model Theory*, p. 150): "I must add that I see little future for model theory without the axiom of choice."

Question: Could there be some interesting choice-free model theory using possibility models instead of Tarskian models?

Partial answer already: Yes, see Guillaume Massas, "A Semi-Constructive Approach to the Hyperreal Line."