Possibilities in Time

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Outline

1. Possibility semantics for tense logic
    1.1 Intervals vs. instants
    1.2 Open future
2. Adding a determinacy operator

Part 2 is based on Cariani's "Modeling Future Indeterminacy in Possibility Semantics."

Possibility semantics for tense logic

Possibility Semantics for tense logic

Let us now read $\Box$ as "it will always be the case that" and $\Diamond$ as "it will sometime be the case that."

We use possibility frames $\mathcal{F} = (S, \sqsubseteq, R)$ with accessibility relation R, but now...

Think of each $x \in S$ as a **stretch** of time and $\sqsubseteq$ as the **substrech** relation (so an *instant*, if there are any, is a $\sqsubseteq$-minimal stretch).

Then $(S, \sqsubseteq)$ is plausibly a separative poset (because if $x \not\sqsubseteq y$, then there should be a substretch of x that has no substretch in common with y).

In "Intervals and Tenses," Roper argues that propositions about *states and processes* correspond to **regular open** subsets of S as before.

Possibility Semantics for tense logic

We still use possibility frames $\mathcal{F} = (S, \sqsubseteq, R)$ with accessibility relation R, but now...

Think of xRy as "x **begins before** y **does**." Then xRy implies that if $\Box\varphi$ is true throughout x, then φ is true throughout y.

We define "φ is true **throughout** x" just as before, with:

- $\mathcal{M}, x \Vdash \Box\varphi$ iff $\forall y \in R(x)$: $\mathcal{M}, y \Vdash \varphi$;
- $\mathcal{M}, x \Vdash \Diamond\varphi$ iff $\forall x' \sqsubseteq x\ \exists y' \in R(x')$: $\mathcal{M}, y' \Vdash \varphi$.

Possibility Semantics for tense logic

We are thinking of xRy as "x begins before y does."

Previous approaches to interval semantics took the relation "x wholly precedes y" (notation: $x > y$) as primitive instead of R. Each approach has some advantages and disadvantages.

(Advantage of R: simpler semantic clauses. Disadvantage: may need separate relations for past and future modalities.)

But arguably, $x > y$ iff *every substretch of x begins before y does*:

$$x > y \text{ iff } \forall x' \sqsubseteq x : x' R y.$$

Possibility Semantics for tense logic

We are thinking of xRy as "x begins before y does."

Under this interpretation of R, interaction conditions on $\sqsubseteq$ and R sufficient for $\Box A$ to be regular open whenever A is are all intuitively correct:

- if xRy and $x \sqsubseteq x'$, then $x'Ry$;
- if xRy and $y' \sqsubseteq y$, then xRy';
- if xRy, then $\exists x' \sqsubseteq x\ \forall x'' \sqsubseteq x'$: $x''Ry$ (even stronger than before).

Additional conditions also make sense in the temporal context (e.g., R should be irreflexive, transitive), and one can catalogue complete logics matching the extra conditions as usual.

Comparison with Roper's "Intervals and Tenses"

Roper's semantics is very similar, but he works with $>$ (or a non-strict version thereof) as a primitive instead of our R (though in his canonical model construction, he defines $x > y$ as $\forall x' \sqsubseteq x$: $x'Ry$, with R defined as we would define it).

The following, which yields persistence of modal formulas in possibility semantics, holds for $Q = R$ but not for $Q = >$:

- if xQy and $x \sqsubseteq x'$, then $x'Qy$.

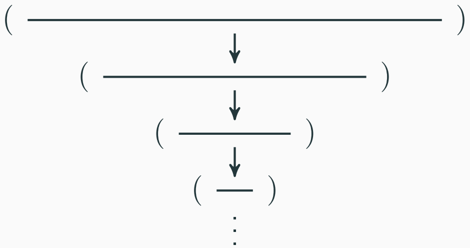
Thus, Roper builds the persistence of modal formulas into his semantic clauses:

- $\mathcal{M}, x \Vdash \Box\varphi$ iff $\forall x' \sqsubseteq x\ \forall y \in Q(x')$: $\mathcal{M}, y \Vdash \varphi$;
- $\mathcal{M}, x \Vdash \Diamond\varphi$ iff $\forall x' \sqsubseteq x\ \exists x'' \sqsubseteq x'\ \exists y \in Q(x'')$: $\mathcal{M}, y \Vdash \varphi$.

Possibility semantics has one fewer quantifier in each clause.

Intervals vs. instants

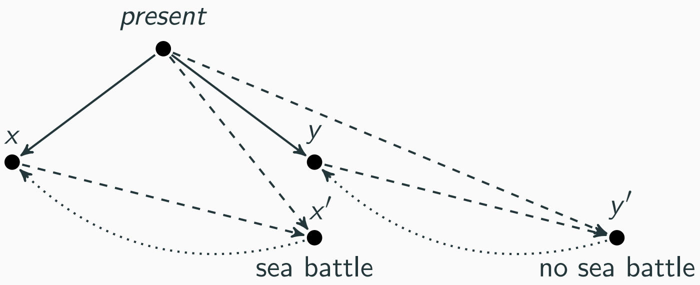
Consider the set $S = \{(a, b) \mid a, b \in \mathbb{Q}, a < b\}$ of all nonempty open intervals $(a, b) = \{x \in \mathbb{Q} \mid a < x < b\}$ of rational numbers. Let $\sqsubseteq$ be the inclusion order: $(a, b) \sqsubseteq (c, d)$ if $(a, b) \subseteq (c, d)$. Thus, we obtain infinite sequences of refinements from infinite chains of shrinking intervals:



Adopting the temporal interpretation of the poset $(S, \sqsubseteq)$, we may think of a possibility as settling that we are now temporally located in some stretch (or “period” or “region”) of time. There is no possibility of a sharpest localization, i.e., no “instants.”

Finally, let $(a, b) R_f (c, d)$ if $a < c$. Then the stated R-$\sqsubseteq$ interaction conditions hold.

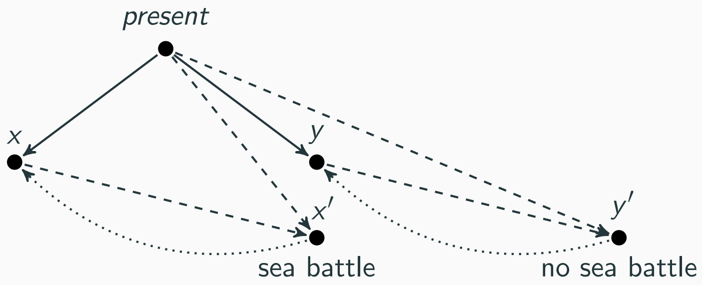
Open future



Example 5.3.9 from "Possibility Semantics." Solid lines are for refinement, dashed for future accessibility, and dotted for past accessibility.

There are two possible refinements of the present, x and y, but we suppose that neither is currently realized.

There are two associated future possibilities for what happens tomorrow: one (x') in which there is sea battle, and one (y') in which there is no sea battle.



Of course, $\mathcal{M}, present \Vdash \Diamond_f sb \vee \neg\Diamond_f sb$. However, $\mathcal{M}, present \nVdash \Diamond_f sb$, since $y \sqsubseteq present$ and $\mathcal{M}, y \Vdash \neg\Diamond_f sb$, and $\mathcal{M}, present \nVdash \neg\Diamond_f sb$, since $x \sqsubseteq present$ and $\mathcal{M}, x \Vdash \Diamond_f sb$. Thus, the future is presently open.

Yet if there is a sea battle, so x' is realized, then the past will turn out to be x, in which there would be a future sea battle, whereas if there is no sea battle, so y' is realized, then the past will turn out to be y, in which there would be no future sea battle. Come tomorrow, we might say, "the past is not what it used to be."

Adding a determinacy operator

Determinacy in the object language

We have seen we can talk in the metalanguage about the openness of the future—the fact that the present did not settle $\Diamond_f sb$ and did not settle $\neg\Diamond_f sb$.

But what if we want to express this in the object language?

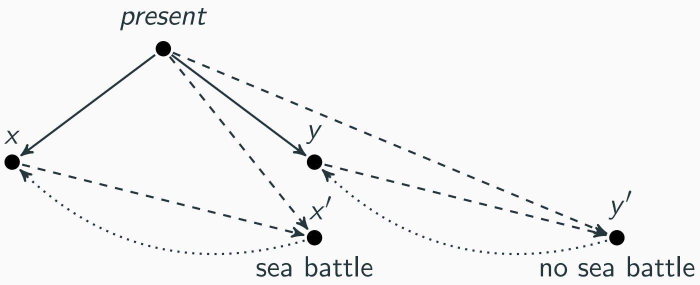
Cariani tackles this in "Modeling future indeterminacy in possibility semantics."

Determinacy in the object language

Cariani argues that we cannot add determinacy operators without moving to a two-dimensional semantics, which works as follows (switching the order of his x, y):

- $\mathcal{M}, x, y \Vdash p$ iff $y \in V(p)$;
- $\mathcal{M}, x, y \Vdash \neg\varphi$ iff $\forall y' \sqsubseteq y$ $\mathcal{M}, x, y' \nVdash \varphi$;
- $\mathcal{M}, x, y \Vdash \varphi \wedge \psi$ iff $\mathcal{M}, x, y \Vdash \varphi$ and $\mathcal{M}, x, y \Vdash \psi$;
- $\mathcal{M}, x, y \Vdash \Box\varphi$ iff for all $y' \in R(y)$, $\mathcal{M}, x, y' \Vdash \varphi$;
- $\mathcal{M}, x, y \Vdash D\varphi$ iff $\mathcal{M}, x, x \Vdash \varphi$.

Finally, define $\mathcal{M}, x \Vdash \varphi$ to mean $\mathcal{M}, x, x \Vdash \varphi$.



Now observe that $\mathcal{M}, \mathit{present} \Vdash \neg D \Diamond_f sb$:

$$
\begin{aligned}
\mathcal{M}, \mathit{present} \Vdash \neg D \Diamond_f sb \quad & \Leftrightarrow \quad \mathcal{M}, \mathit{present}, \mathit{present} \Vdash \neg D \Diamond_f sb \\
& \Leftrightarrow \quad \forall z \sqsubseteq \mathit{present}, \mathcal{M}, \mathit{present}, z \nVdash D \Diamond_f sb \\
& \Leftrightarrow \quad \forall z \sqsubseteq \mathit{present}, \mathcal{M}, \mathit{present}, \mathit{present} \nVdash \Diamond_f sb \\
& \Leftrightarrow \quad \mathcal{M}, \mathit{present}, \mathit{present} \nVdash \Diamond_f sb
\end{aligned}
$$

General fact: $\mathcal{M}, x \Vdash \neg D\varphi$ iff $\mathcal{M}, x \nVdash \varphi$.

Going outside $\mathcal{RO}(S, \sqsubseteq)$

Let $[\![\varphi]\!]^{\mathcal{M}} = \{x \in S \mid \mathcal{M}, x \Vdash \varphi\}$.

For the language without D, $[\![\varphi]\!]^{\mathcal{M}}$ is always regular open.

But this is no longer true when D is allowed. Persistence and refinability fail:

- From $\mathcal{M}, x \Vdash \neg Dq$ and $x' \sqsubseteq x$, we cannot conclude $\mathcal{M}, x' \Vdash \neg Dq$;
- From $\mathcal{M}, x \nVdash Dq$, we cannot conclude that there is an $x' \sqsubseteq x$ such that for all $x'' \sqsubseteq x'$, $\mathcal{M}, x'' \nVdash Dq$.

So D takes us outside $\mathcal{RO}(S, \sqsubseteq)$. But that's okay if we weren't expecting to get classical logic with D in the language. What is the logic with D in the language?

Example of non-classicality: $[\![q]\!]^{\mathcal{M}} \subseteq [\![Dq]\!]^{\mathcal{M}}$ but $[\![\neg Dq]\!]^{\mathcal{M}} \not\subseteq [\![\neg q]\!]^{\mathcal{M}}$.

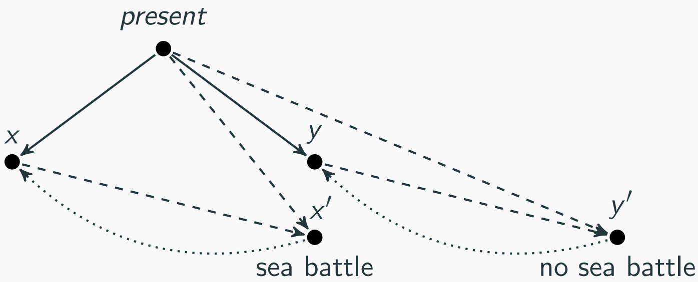
Comparison with inquisitive disjunction

Let's compare the D operator with *inquisitive disjunction*, viewing inquisitive semantics from the point of view of possibility semantics as in Section 8 of "Possibility frames and forcing for modal logic" or Section 1 of "Inquisitive intuitionistic logic."

We add to our language (without D) a binary connective $\vee\!\!\!\vee$ with the following clause:

- $\mathcal{M}, x \Vdash \varphi \vee\!\!\!\vee \psi$ iff $\mathcal{M}, x \Vdash \varphi$ or $\mathcal{M}, x \Vdash \psi$.

Could we take $\mathcal{M}, x \Vdash \varphi \vee\!\!\!\vee \neg\varphi$ to mean that *it is determinate whether* φ *holds at* x?



Since $\mathcal{M}, \textit{present} \not\Vdash \Diamond_f sb$ and $\mathcal{M}, \textit{present} \not\Vdash \neg\Diamond_f sb$, we have $\mathcal{M}, \textit{present} \not\Vdash \Diamond_f sb \mathbin{\vee\!\!\vee} \neg\Diamond_f sb$.

Thus, at present, it is not determinate whether there will be a sea battle.

But notice how the ‘not’ in ‘not determinate’ remains at the level of the metalanguage—a point to which we’ll return shortly. . .

Differences

Though $\mathbin{\vee\!\!\!\vee}$ takes us outside $\mathcal{RO}(S, \sqsubseteq)$, since refinability fails for $p \mathbin{\vee\!\!\!\vee} q$, we still have persistence, so we stay inside the Heyting algebra $\mathrm{Down}(S, \sqsubseteq)$ of all downsets.

This is why inquisitive logic is like a (super)intuitionistic logic, plus the special axiom $\neg\neg q \to q$ for atomic sentences, since the semantics evaluates them in $\mathcal{RO}(S, \sqsubseteq)$.

Another big difference comes when trying to express *indeterminacy*: compare $\neg Dq \wedge \neg D\neg q$ and $\neg(q \mathbin{\vee\!\!\!\vee} \neg q)$. Unlike the former, the latter is unsatisfiable! The move to two-dimensional semantics for D allows the former to be satisfiable.