Comparative Probability

ESSLLI 2014 - Logic and Probability

Wes Holliday and Thomas Icard
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Outline

1. Basic Notions
2. Considering Axioms: Rationality and Realism
   • de Finetti’s Axioms
3. Comparative Probability as Modal Logic
4. Probabilistic Representability
   • The Kraft-Pratt-Seidenberg Counterexample
   • Cancellation Axioms and Scott’s Representation Theorem
5. Segerberg and Gardenfors’ Completeness Theorem
6. Bonus 1: Imprecise Representability (by a set of measures)
7. Bonus 2: Extending an Ordering on a Set to the Powerset
References for comparative probability:


References for comparative probability as modal logic:

Given a set $W$, we will consider a binary relation $\succeq$ on $\mathcal{P}(W)$.

For “events” or “propositions” $A, B \in \mathcal{P}(W)$, our interpretation is:

$A \succeq B$ – “$A$ is (regarded by the agent as) at least as probable as $B$.”

To be more general, we could take $\succeq$ to be a binary relation on some algebra $\mathcal{A}$ of subsets of $W$, not necessarily $\mathcal{P}(W)$; but since we’ll mostly focus on the case of finite $W$, this won’t be necessary.
Non-strict Relation as Basic

$A \succeq B$ – "$A$ is (regarded by the agent as) at least as probable as $B$";

Define: $A \succ B$ iff $A \succeq B$ and $B \not\succeq A$;

Define: $A \sim B$ iff $A \succeq B$ and $B \succeq A$. This indicates that "$A$ and $B$ are (regarded by the agent as) equally probable."

Define: $A \uparrow B$ iff $A \not\succeq B$ and $B \not\succeq A$. This indicates that "$A$ and $B$ are incomparable in probability (for the agent)."

NB: $A \succ B$ doesn’t necessarily indicate “$A$ is (regarded by the agent as) more probable than $B$.“ One may judge that $A$ is at least as likely as $B$, but withhold judgment on whether $B$ is at least as likely as $A$. That’s not to judge that $A$ is more probable.
Basic Notions

Strict Relation as Basic

\[ A > B \text{ – “} A \text{ is (regarded by the agent as) more probable than } B. \text{”} \]

Define: \( A \approx B \) iff \( A \not> B \) and \( B \not> A \);

Define: \( A \succeq B \) either \( A > B \) or \( A \approx B \).

**NB:** \( A \approx B \) doesn’t necessarily indicate that “\( A \text{ and } B \text{ are equally probable} \)” i.e., \( A \sim B \). It may be that the events are incomparable for the agent. That’s not the same as being equally probable.
Basic Notions

Strict Relation as Basic

\( A > B \) – “\( A \) is (regarded by the agent as) more probable than \( B \).”

Define: \( A \approx B \) iff \( A \not> B \) and \( B \not> A \);

Define: \( A \succeq B \) either \( A > B \) or \( A \approx B \).

NB: even if we assume non-strict completeness, i.e., for all \( A, B \),
“\( A \) is at least as probable as \( B \) or \( B \) is at least as probable as \( A \),”
\( A \approx B \) still doesn’t mean that “\( A \) and \( B \) are equally probable.”

It may be that the agent judges that \( A \) is at least as probable as \( B \),
but withholds judgment on whether \( A \) is more probable than \( B \) and
withholds judgment on whether \( B \) is more probable than \( A \). Then
\( A \approx B \), but the agent does not judge them to be equally probable.

It follows that \( A \succeq B \) doesn’t necessarily indicate that \( A \succeq B \).
Basic Notions

Non-strict as basic:

\( A \succsim B \) – “\( A \) is at least as probable as \( B \)”;

Define: \( A \succ B \) iff \( A \succsim B \) and \( B \nleq A \);
Define: \( A \sim B \) iff \( A \succsim B \) and \( B \succsim A \). “\( A \) and \( B \) are equally probable.”

Strict as basic:

\( A > B \) – “\( A \) is more probable than \( B \)”.
Define: \( A \approx B \) iff \( A \nprec B \) and \( B \nprec A \);
Define: \( A \succeq B \) either \( A > B \) or \( A \approx B \).

The **moral** of the foregoing seems to be that if we want to capture both “at least as probable as” and “more probable than,” then we should take both \( \succsim \) and \( > \) as primitive, related by the axioms:

- if \( A > B \), then \( A \succeq B \);
- if \( A \succeq B \), then \( B \npreceq A \).

**However**, typically just one of \( \succsim \) or \( > \) is taken as primitive...
Basic Notions

Non-strict as basic:

\( A \preccurlyeq B \) – “\( A \) is at least as probable as \( B \)”;

Define: \( A \succ B \) iff \( A \preccurlyeq B \) and \( B \npreccurlyeq A \);

Define: \( A \sim B \) iff \( A \preccurlyeq B \) and \( B \preccurlyeq A \). “\( A \) and \( B \) are equally probable.”

Strict as basic:

\( A \succ B \) – “\( A \) is more probable than \( B \)”.

Define: \( A \approx B \) iff \( A \not\succ B \) and \( B \not\succ A \);

Define: \( A \succeq B \) either \( A \succ B \) or \( A \approx B \).

Exercise

If \( \preccurlyeq \) is basic and \( \succ \) and \( \sim \) are defined as above, then the following equivalences (from the def.’s in terms of \( \succ \)) hold iff \( \preccurlyeq \) is complete:

1. \( A \sim B \) iff \( A \npreccurlyeq B \) and \( B \npreccurlyeq A \);

2. \( A \succeq B \) iff \( A \succ B \) or \( A \sim B \).
Basic Notions

Non-strict as basic:

\(A \succeq B\) – “\(A\) is at least as probable as \(B\)”;
Define: \(A \succ B\) iff \(A \succeq B\) and \(B \not\succeq A\);
Define: \(A \sim B\) iff \(A \succeq B\) and \(B \succeq A\). “\(A\) and \(B\) are equally probable.”

Strict as basic:

\(A > B\) – “\(A\) is more probable than \(B\)”.
Define: \(A \approx B\) iff \(A \not> B\) and \(B \not> A\);
Define: \(A \preceq B\) either \(A > B\) or \(A \approx B\).

Exercise

If \(>\) is basic and \(\approx\) and \(\succeq\) defined as above, then the following (from the def.’s in terms of \(\succeq\)) hold iff \(>\) is asymmetric:

1. \(A > B\) iff \(A \succeq B\) and \(B \not\succeq A\);
2. \(A \approx B\) iff \(A \succeq B\) and \(B \succeq A\).
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What Axioms are Evident?

Since $\succeq$ means “at least as probable,” we should at least have:

- **reflexivity**: $A \succeq A$.

Fishburn: the following might count as “so obvious and uncontroversial as to occasion no serious criticism.”

- **nonnegativity**: $A \succeq \emptyset$;
- **nontriviality**: $\emptyset \not\succeq W$ (or $W > \emptyset$);
- **monotonicity**: if $A \supseteq B$, then $A \succeq B$;
- **asymmetry**: $A > B$ implies $B \nprec A$; (The asymmetry of $\succ$ follows from the fact that $A \succ B := [A \succeq B \text{ and } B \nprec A]$.)
- **inclusion monotonicity**: if $[A \supseteq B \text{ and } B > C]$ or $[A > B \text{ and } B \supseteq C]$, then $A > C$. (If $\succeq$ is transitive and monotonic, then inclusion monotonicity holds for $\succ$.)
Fishburn: one might also suggest the following axioms as obvious, but they have been challenged.

- **transitivity**: if \( A > B \) and \( B > C \), then \( A > C \); similarly for \( \sim \). (Exercise: what axioms give you transitivity for \( \succ \)?)

- **complementarity**: if \( A \succ B \), then \( W - B \ncc W - A \). (We will also consider **self-duality**: \( A \sim B \) iff \( W - B \sim W - A \).)

- **quasi-additivity**: if \( (A \cup B) \cap C = \emptyset \), then \( A \cc B \iff A \cup C \cc B \cup C \).
de Finetti’s Axioms

de Finetti’s proposed the following axioms for \( \succcurlyeq \):

- **nonnegativity**: \( A \succcurlyeq \emptyset \);
- **nontriviality**: \( \emptyset \not\succcurlyeq W \);
- **totality**: \( A \succcurlyeq B \) of \( B \succcurlyeq A \);
- **transitivity**: if \( A \succcurlyeq B \) and \( B \succcurlyeq C \), then \( A \succcurlyeq C \);
- **quasi-additivity**: if \( (A \cup B) \cap C = \emptyset \), then \( A \succcurlyeq B \iff A \cup C \succcurlyeq B \cup C \).
de Finetti’s Axioms

de Finetti’s proposed that \( \succsim \) is a total preorder satisfying:

- **nonnegativity**: \( A \succsim \emptyset \);
- **nontriviality**: \( \emptyset \not\succsim W \);
- **quasi-additivity**: if \( (A \cup B) \cap C = \emptyset \), then \( A \succsim B \iff A \cup C \succsim B \cup C \).

An equivalent formulation replaces the last axiom with:

- \( A \succsim B \) iff \( A - B \succsim B - A \).

**Exercise**

Show that if \( \succsim \) satisfies de Finetti’s axioms, then it also satisfies **monotonicity** and **self-duality**.
Doubts about Totality

One of de Finetti’s axioms is:

- **totality**: $A \simeq B$ or $B \simeq A$.

As Fine (1973, 18) notes, “the requirement that all events be comparable is not insignificant and has been denied by many careful students of probability including Keynes and Koopman.”

For example:

- $A$ a leaf will fall on my shoe on the walk down to town.
- $B$ one of my kitchen appliances will break in the next 24 hours.

Must it be that either I regard $A$ to be at least as likely as $B$ or I regard $B$ to be at least as likely as $A$?
Consider the following for a *slightly bent* coin:

- **A**: The next 101 flips will give at least 40 heads.
- **B**: The next 100 flips will give at least 40 heads.
- **C**: The next 1000 flips will give at least 460 heads.

Are the following judgments unreasonable?

- **A > B**, **A ~ C**, **B ~ C**.

Fishburn says that the judgments are not unreasonable.

According to these judgments, **B ~ C** and **C ~ A**, but **B ~ A**, so **~** is *not transitive*. 
Another Example from Fishburn

Sue is expecting to meet Smith. She doesn’t know his age or appearance. Her judgments about the likelihood of his attributes:

- **Height**: 6’-0” > 6’-1” > 6’-2’’;
- **Age**: 40 > 50 > 60;
- **Hair**: bown > red > blonde.

Three possible combinations are:

A 6′-0″ 60-year old redhead;
B 6′-1″ 40-year old blonde;
C 6′-2″ 50-year old brunette.

Sue judges a combination $X$ more likely than combination $Y$ if $X$ is more probable than $Y$ on two of the attributes.

Thus, $A > B$, $B > C$, and $C > A$, so we have intransitivity.
Ellsberg 1961

A bag contains 90 marbles. It is known that 30 of the marbles are red. It is also known that each of the remaining 60 marbles is either green or purple, but the proportion is not known.

You’ll draw a marble at random. It seems reasonable to assign probability $\frac{3}{9}$ to drawing a red marble ($R$). It also seems reasonable to assign probability $\frac{6}{9}$ to drawing either a green or purple one ($G \cup P$). But what about the probabilities of $G$ and $P$?
A bag contains 90 marbles. It is known that 30 of the marbles are red. It is also known that each of the remaining 60 marbles is either green or purple, but the proportion is not known.

For each event $E$ consider the following bet:

- $B_E$: pays 1 Euro if $E$; 0 otherwise.

Which do you prefer: $B_R$ or $B_G$? $B_R$ or $B_P$? $B_{G\cup P}$ or $B_{R\cup P}$?
Ellsberg 1961

A bag contains 90 marbles. It is known that 30 of the marbles are red. It is also known that each of the remaining 60 marbles is either green or purple, but the proportion is not known.

For each event $E$ consider the following bet:

- $B_E$: pays 1 Euro if $E$; 0 otherwise.

**Empirically**: people tend to strictly prefer $B_R$ to $B_G$ (and to $B_P$), and they tend to prefer $B_{G\cup P}$ to $B_{R\cup P}$.

**Moral**: if the preferences reflect comparative probability judgments (e.g., if the people are expected utility maximizers), then the judgments violate quasi-additivity:

\[ G \cup P \succsim R \cup P, \text{ but } G \not\succsim R. \]
Let’s assume that the foregoing examples can be explained away.

So let’s assume that de Finetti was correct that \( \succeq \) should be a total preorder satisfying:

- **nonnegativity**: \( A \succeq \emptyset \);
- **nontriviality**: \( \emptyset \not\succeq W \);
- **quasi-additivity**: if \( (A \cup B) \cap C = \emptyset \), then \( A \succeq B \iff A \cup C \succeq B \cup C \).
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As Segerberg and later Gärdenfors observed, comparative probability can be seen as a kind of modal logic.

In this spirit, we’ll introduce a formal modal language...
Formal Language

Definition

Given a countable set \( \text{At} = \{p, q, r, \ldots\} \) of atomic sentences, the language \( \mathcal{L}(\Box, \geq) \) is defined by the grammar:

\[
\varphi ::= p \mid \neg \varphi \mid (\varphi \land \varphi) \mid \Box \varphi \mid (\varphi \geq \psi).
\]

We define \( \lor, \rightarrow, \) and \( \leftrightarrow \) as usual, and

\[
\begin{align*}
\top & ::= (p \lor \neg p) \text{ for some } p \in \text{At}; \\
\bot & ::= \neg \top; \\
\Diamond \varphi & ::= \neg \Box \neg \varphi; \\
(\varphi > \psi) & ::= (\varphi \geq \psi) \land \neg (\psi \geq \varphi); \\
\triangle \varphi & ::= (\varphi > \neg \varphi).
\end{align*}
\]
Models

Definition

A QP Kripke model for $\mathcal{L}(\Box, \succeq)$ is a tuple $\mathcal{M} = \langle W, R, \{\succsim_w\}_{w \in W}, V \rangle$ where:

1. $W$ is a nonempty set;
2. $R$ is a serial binary relation on $W$, i.e., such that for all $w \in W$, $R(w) = \{v \in W \mid wRv\} \neq \emptyset$;
3. $\succsim_w$ is a binary relation on $\powerset(R(w))$;
4. $V : \text{At} \rightarrow \powerset(W)$.

NB: Segerberg uses a more general definition where $\succsim_w$ is a binary relation on some algebra of subsets of $R(w)$, not necessarily $\powerset(W)$. Since we’ll focus on finite models, this won’t matter.
Truth

Definition
Given a QP Kripke model $\mathcal{M}$ and a $\varphi \in \mathcal{L}(\Box, \succeq)$, we define $\mathcal{M}, w \models \varphi$ ("$\varphi$ is true at state $w$ in model $\mathcal{M}$") as follows:

1. $\mathcal{M}, w \models p$ iff $w \in V(p)$;
2. $\mathcal{M}, w \models \neg \varphi$ iff $\mathcal{M}, w \not\models \varphi$;
3. $\mathcal{M}, w \models (\varphi \land \psi)$ iff $\mathcal{M}, w \models \varphi$ and $\mathcal{M}, w \models \psi$;
4. $\mathcal{M}, w \models \Box \varphi$ iff $\forall v \in R(w): \mathcal{M}, v \models \varphi$;
5. $\mathcal{M}, w \models \varphi \succeq \psi$ iff $[\varphi]_w^{\mathcal{M}} \succeq_w [\psi]_w^{\mathcal{M}}$,

where $[\chi]_w^\mathcal{M} = \{v \in R(w) \mid \mathcal{M}, v \models \chi\}$. 
Yalcin’s List of Intuitively Valid and Invalid Patterns

V1  $\triangledown \varphi \rightarrow \neg \triangledown \neg \varphi$
V2  $\triangledown (\varphi \land \psi) \rightarrow (\triangledown \varphi \land \triangledown \psi)$
V3  $\triangledown \varphi \rightarrow \triangledown (\varphi \lor \psi)$
V4  $\varphi \geq \bot$
V5  $\top \geq \varphi$
V6  $\Box \varphi \rightarrow \triangle \varphi$
V7  $\triangle \varphi \rightarrow \lozenge \varphi$
V11  $(\psi \geq \varphi) \rightarrow (\triangledown \varphi \rightarrow \triangledown \psi)$
V12  $(\psi \geq \varphi) \rightarrow ((\varphi \geq \neg \varphi) \rightarrow (\psi \geq \neg \psi))$
I1  $((\varphi \geq \psi) \land (\varphi \geq \chi)) \rightarrow (\varphi \geq (\psi \lor \chi))$
I2  $(\varphi \geq \neg \varphi) \rightarrow (\varphi \geq \psi)$
I3  $\triangledown \varphi \rightarrow (\varphi \geq \psi)$
E1  $(\triangledown \varphi \land \triangledown \psi) \rightarrow \triangledown (\varphi \land \psi)$
System W

Taut all tautologies

\[ \frac{\varphi}{\Box \varphi} \]

MP \[ \frac{\varphi \to \psi}{\psi} \]

Nec \[ \frac{\varphi}{\Box \varphi} \]

K \[ \square (\varphi \to \psi) \to (\square \varphi \to \square \psi) \]

Ex \[ (\square (\varphi \leftrightarrow \varphi') \wedge \square (\psi \leftrightarrow \psi')) \to ((\varphi \geq \psi) \leftrightarrow (\varphi' \geq \psi')) \]

Bot \[ \varphi \geq \bot \]

BT \[ \neg (\bot \geq \top) \]

Tran \[ (\varphi \geq \psi) \to ((\psi \geq \chi) \to (\varphi \geq \chi)) \]

Mon \[ \square (\varphi \to \psi) \to (\varphi \geq \psi) \]
Comparative Probability as Modal Logic

Taut all tautologies

MP \[ \phi \rightarrow \psi \vdash \phi \]

Nec \[ \phi \vdash \Box \phi \]

K \[ \Box (\phi \rightarrow \psi) \rightarrow (\Box \phi \rightarrow \Box \psi) \]

Ex \[ (\Box (\phi \leftrightarrow \phi') \land \Box (\psi \leftrightarrow \psi')) \rightarrow ((\phi \geq \psi) \leftrightarrow (\phi' \geq \psi')) \]

Bot \[ \phi \geq \bot \]

BT \[ \neg (\bot \geq \top) \]

Tran \[ (\phi \geq \psi) \rightarrow ((\psi \geq \chi) \rightarrow (\phi \geq \chi)) \]

Mon \[ \Box (\phi \rightarrow \psi) \rightarrow (\phi \geq \psi) \]

Fact (Holliday and Icard 2013)

All of Yalcin’s V1-V12, and none of I1-I3 or E1, are derivable in the system WS that adds to W the following axiom:

S \[ (\phi \geq \psi) \rightarrow (\neg \psi \geq \neg \phi). \]
Yalcin’s List of Intuitively Valid and Invalid Patterns

V1 \( \triangle \varphi \rightarrow \neg \triangle \neg \varphi \)
V2 \( \triangle (\varphi \land \psi) \rightarrow (\triangle \varphi \land \triangle \psi) \)
V3 \( \triangle \varphi \rightarrow \triangle (\varphi \lor \psi) \)
V4 \( \varphi \geq \bot \)
V5 \( \top \geq \varphi \)
V6 \( \Box \varphi \rightarrow \triangle \varphi \)
V7 \( \triangle \varphi \rightarrow \Diamond \varphi \)
V11 \( (\psi \geq \varphi) \rightarrow (\triangle \varphi \rightarrow \triangle \psi) \)
V12 \( (\psi \geq \varphi) \rightarrow ((\varphi \geq \neg \varphi) \rightarrow (\psi \geq \neg \psi)) \)

I1 \( ((\varphi \geq \psi) \land (\varphi \geq \chi)) \rightarrow (\varphi \geq (\psi \lor \chi)) \)
I2 \( \varphi \geq \neg \varphi \rightarrow (\varphi \geq \psi) \)
I3 \( \triangle \varphi \rightarrow (\varphi \geq \psi) \)
E1 \( (\triangle \varphi \land \triangle \psi) \rightarrow \triangle (\varphi \land \psi) \)
de Finetti’s Axioms

de Finetti’s proposed the following axioms for $\succsim$:

- **nonnegativity**: $A \succsim \emptyset$;
- **nontriviality**: $\emptyset \nsuccsim W$;
- **totality**: $A \succsim B$ of $B \succsim A$;
- **transitivity**: if $A \succsim B$ and $B \succsim C$, then $A \succsim C$;
- **quasi-additivity**: if $(A \cup B) \cap C = \emptyset$, then $A \succsim B \iff A \cup C \succsim B \cup C$. 
### de Finetti’s System \textbf{FA}

<table>
<thead>
<tr>
<th>Rule</th>
<th>Formula</th>
</tr>
</thead>
<tbody>
<tr>
<td>Taut</td>
<td>all tautologies</td>
</tr>
<tr>
<td>MP</td>
<td>$\frac{\varphi \rightarrow \psi}{\psi}$</td>
</tr>
<tr>
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</tr>
<tr>
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</tr>
<tr>
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<td>$(\Box (\varphi \leftrightarrow \varphi') \land \Box (\psi \leftrightarrow \psi')) \rightarrow ((\varphi \geq \psi) \leftrightarrow (\varphi' \geq \psi'))$</td>
</tr>
<tr>
<td>Bot</td>
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</tr>
<tr>
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<td>$\neg (\bot \geq \top)$</td>
</tr>
<tr>
<td>Tot</td>
<td>$(\varphi \geq \psi) \lor (\psi \geq \varphi)$</td>
</tr>
<tr>
<td>Tran</td>
<td>$(\varphi \geq \psi) \rightarrow ((\psi \geq \chi) \rightarrow (\varphi \geq \chi))$</td>
</tr>
<tr>
<td>A</td>
<td>$\neg \Box ((\varphi \lor \psi) \land \chi) \rightarrow (\varphi \geq \psi \leftrightarrow ((\varphi \lor \chi) \geq (\psi \lor \chi)))$</td>
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Let $\mathcal{A}$ be an algebra on a nonempty set $W$, e.g., $\mathcal{A}$ is $\mathcal{P}(W)$.

**Definition (Agreement)**

Given a binary relation $\succeq$ on $\mathcal{A}$ and a probability measure $\mu : \mathcal{A} \to [0, 1]$, $\mu$ agrees with $\succeq$ iff for all $X, Y \in \mathcal{A}$:

$$X \succeq Y \iff \mu(X) \geq \mu(Y).$$

We also say that $\mu$ represents $\succeq$.

**Definition (Representability)**

A binary relation $\succeq$ on $\mathcal{P}(W)$ is probabilistically representable iff there is a probability measure $\mu$ on $\mathcal{P}(W)$ that agrees with $\succeq$.

**QUESTION**: is any relation $\succeq$ that satisfies de Finetti’s three axioms probabilistically representable?
Answer: No!

Fact (Kraft, Pratt, and Seidenberg)
There is a total preorder $\succeq$ on $\wp(\{a, b, c, d, e\})$ such that:

1. $\succeq$ is nontrivial: $\emptyset \not\succeq W$;
2. $\succeq$ is nonnegative: $A \succeq \emptyset$;
3. $\succeq$ is quasi-additive: if $(A \cup B) \cap C = \emptyset$, then $A \succeq B \iff A \cup C \succeq B \cup C$;
4. $\succeq$ is not probabilistically representable.

Let’s do the proof, expanding the presentation of Krantz et al.
Proof. Suppose there is a $\succ$ satisfying de Finetti’s axioms and:

$\{a\} \succ \{b, c\}$  $\{c, d\} \succ \{a, b\}$  $\{b, e\} \succ \{a, c\}$  $\{a, b, c\} \succ \{d, e\}$.

Then $\succ$ isn’t representable. For if there were an agreeing $\mu$, then

$\mu(\{a\}) > \mu(\{b, c\})$  $\mu(\{c, d\}) > \mu(\{a, b\})$  $\mu(\{b, e\}) > \mu(\{a, c\}),$

so by finite additivity of $\mu$,

$$
\begin{align*}
\mu(a) & > \mu(b) + \mu(c) \\
\mu(c) + \mu(d) & > \mu(a) + \mu(b) \\
\mu(b) + \mu(e) & > \mu(a) + \mu(c).
\end{align*}
$$

Adding the left and right sides of these inequalities,

$$
\mu(a) + \mu(b) + \mu(c) + \mu(d) + \mu(e) > 2\mu(a) + 2\mu(b) + 2\mu(c).
$$
Proof. Suppose there is a $\succ$ satisfying de Finetti’s axioms and:

$$\{a\} \succ \{b, c\} \quad \{c, d\} \succ \{a, b\} \quad \{b, e\} \succ \{a, c\} \quad \{a, b, c\} \succ \{d, e\}.$$  

Then $\succ$ isn’t representable. For if there were an agreeing $\mu$, then

$$\mu(\{a\}) > \mu(\{b, c\}) \quad \mu(\{c, d\}) > \mu(\{a, b\}) \quad \mu(\{b, e\}) > \mu(\{a, c\}),$$

so by finite additivity of $\mu$,

$$\mu(a) > \mu(b) + \mu(c)$$
$$\mu(c) + \mu(d) > \mu(a) + \mu(b)$$
$$\mu(b) + \mu(e) > \mu(a) + \mu(c).$$

Adding the left and right sides of these inequalities,

$$\mu(a) + \mu(b) + \mu(c) + \mu(d) + \mu(e) > 2\mu(a) + 2\mu(b) + 2\mu(c).$$
Proof. Suppose there is a $\succsim$ satisfying de Finetti’s axioms and:

$\{a\} \succ \{b, c\} \quad \{c, d\} \succ \{a, b\} \quad \{b, e\} \succ \{a, c\} \quad \{a, b, c\} \succ \{d, e\}.$

Then $\succsim$ isn’t representable. For if there were an agreeing $\mu$, then

$\mu(\{a\}) > \mu(\{b, c\}) \quad \mu(\{c, d\}) > \mu(\{a, b\}) \quad \mu(\{b, e\}) > \mu(\{a, c\}),$

so by finite additivity of $\mu$,

\[
\mu(a) > \mu(b) + \mu(c) \\
\mu(c) + \mu(d) > \mu(a) + \mu(b) \\
\mu(b) + \mu(e) > \mu(a) + \mu(c).
\]

Adding the left and right sides of these inequalities,

\[
\mu(d) + \mu(e) > \mu(a) + \mu(b) + \mu(c).
\]

Then by the assumption that $\mu$ agrees with $\succsim$, we would have $\{d, e\} \succ \{a, b, c\}$, contradicting $\{a, b, c\} \succ \{d, e\}$. 
Proof. \textit{Suppose} there is a $\succeq$ satisfying de Finetti’s axioms and:

\[
\{a\} \succeq \{b, c\} \quad \{c, d\} \succeq \{a, b\} \quad \{b, e\} \succeq \{a, c\} \quad \{a, b, c\} \succeq \{d, e\}.
\]

Then $\succeq$ isn’t representable. ✓ To show that there \textit{is} such a relation $\succeq$, for some $0 < \epsilon < 1/3$, define

\[
\nu(a) = 4 - \epsilon \quad \nu(b) = 1 - \epsilon \quad \nu(c) = 3 - \epsilon \quad \nu(d) = 2 \quad \nu(e) = 6,
\]

\[
\nu(A) = \sum_{a \in A} \nu(a), \text{ which can be normalized as } \mu(A) = \frac{\nu(A)}{16 - 3\epsilon}.
\]

Define $\succeq_\nu$ s.th. $A \succeq_\nu B$ iff $\nu(A) \geq \nu(B)$ (iff $\mu(A) \geq \mu(B)$).

Thus, $\succeq_\nu$ satisfies de Finetti’s axioms, and the \textcolor{blue}{blue} $\succ$’s hold:

\[
\nu(\{a\}) = 4 - \epsilon \quad \nu(\{b, c\}) = 4 - 2\epsilon \\
\nu(\{c, d\}) = 5 - \epsilon \quad \nu(\{a, b\}) = 5 - 2\epsilon \\
\nu(\{b, e\}) = 7 - \epsilon \quad \nu(\{a, c\}) = 7 - 2\epsilon.
\]
Proof. *Suppose* there is a $\succeq$ satisfying de Finetti’s axioms and:

$$\{a\} \succeq \{b, c\} \quad \{c, d\} \succeq \{a, b\} \quad \{b, e\} \succeq \{a, c\} \quad \{a, b, c\} \succeq \{d, e\}.$$  

Then $\succeq$ isn’t representable. For some $0 < \epsilon < 1/3$, define

$$\nu(a) = 4 - \epsilon \quad \nu(b) = 1 - \epsilon \quad \nu(c) = 3 - \epsilon \quad \nu(d) = 2 \quad \nu(e) = 6.$$  

Define $\succeq_\nu$ s.th. $A \succeq_\nu B$ iff $\nu(A) \geq \nu(B)$ (iff $\mu(A) \geq \mu(B)$).

Thus, $\succeq_\nu$ satisfies de Finetti’s axioms, and the blue $\succ$’s hold. ✓

Let $\succeq'_\nu = \succeq_\nu - \{\langle\{d, e\}, \{a, b, c\}\rangle\} \cup \{\langle\{a, b, c\}, \{d, e\}\rangle\}$, so blue $\succ$’s and red $\succ$ hold for $\succeq'_\nu$. **Claim:** deFi’s axioms hold of $\succeq'_\nu$.

$$\nu(\{d, e\}) = 8, \nu(\{a, b, c\}) = 8 - 3\epsilon,$$  

but there is no distinct $A$ with $8 \geq \nu(A) \geq 8 - 3\epsilon$; for it requires $\nu(A) = 8 - k\epsilon$ for $k \in \{0, 1, 2, 3\}$; but 8 can be expressed as a sum of distinct members of $\{1, 2, 3, 4, 6\}$ in only two ways, $4 + 1 + 3$ and $2 + 6$, which pick out $\{a, b, c\}$ and $\{d, e\}$.

Thus, there is no distinct $A$ with $\{d, e\} \succeq_\nu A \succeq_\nu \{a, b, c\}$. 
Proof. Suppose there is a $\succsim$ satisfying de Finetti’s axioms and:

$$\{a\} \succsim \{b, c\} \quad \{c, d\} \succsim \{a, b\} \quad \{b, e\} \succsim \{a, c\} \quad \{a, b, c\} \succsim \{d, e\}.$$  

Let $\succsim'_v = \succsim_v - \{\langle \{d, e\}, \{a, b, c\} \rangle \} \cup \{\langle \{a, b, c\}, \{d, e\} \rangle \}$, so blue $\succ$’s and red $\succsim$ hold for $\succsim'_v$. Claim: deFi’s axioms still hold.

We’ve shown there is no distinct $A$ with $\{d, e\} \succsim_v A \succsim_v \{a, b, c\}$.

Check: $(A \cup B) \cap C = \emptyset \Rightarrow [A \cup C \succsim'_v B \cup C \iff A \succsim'_v B]$.

- If $A = B$ or $A \cup C = B \cup C$, then the principle holds by reflexivity.
- If $A = \{d, e\}$ and $B = \{a, b, c\}$ or vice versa, then $C = \emptyset$, so it holds; similarly if $A \cup C = \{d, e\}$ and $B \cup C = \{a, b, c\}$ or vice versa.

So we can assume $A \notin S = \{\{d, e\}, \{a, b, c\}\}$ or $B \notin S$. Then the purple fact above implies that $A \succsim_v B$ iff $A \succsim'_v B$ (picture it).

We can also assume $A \cup C \notin S$ or $B \cup C \notin S$. Then the purple fact above implies that $A \cup C \succsim_v B \cup C$ iff $A \cup C \succsim'_v B \cup C$.

Thus, the quasi-additivity of $\succsim_v$ implies the quasi-additivity of $\succsim'_v$. 

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Proof. *Suppose* there is a \( \succcurlyeq \) satisfying de Finetti’s axioms and:

\[
\{ a \} \succ \{ b, c \} \quad \{ c, d \} \succ \{ a, b \} \quad \{ b, e \} \succ \{ a, c \} \quad \{ a, b, c \} \succ \{ d, e \}.
\]

Then \( \succcurlyeq \) isn’t representable. ✓ To show that there *is* such a relation \( \succcurlyeq \), for some \( 0 < \epsilon < 1/3 \), define

\[
\nu(a) = 4 - \epsilon \quad \nu(b) = 1 - \epsilon \quad \nu(c) = 3 - \epsilon \quad \nu(d) = 2 \quad \nu(e) = 6,
\]

\[
\nu(A) = \sum_{a \in A} \nu(a), \text{ which can be normalized as } \mu(A) = \frac{\nu(A)}{16 - 3\epsilon}.
\]

Define \( \succcurlyeq_{\nu} \) s.th. \( A \succcurlyeq_{\nu} B \) iff \( \nu(A) \geq \nu(B) \) (iff \( \mu(A) \geq \mu(B) \)).

Thus, \( \succcurlyeq_{\nu} \) satisfies de Finetti’s axioms, and the blue \( \succcurlyeq \)'s hold. ✓

Let \( \succcurlyeq'_{\nu} = \succcurlyeq_{\nu} - \{ \langle \{ d, e \}, \{ a, b, c \} \rangle \} \cup \{ \langle \{ a, b, c \}, \{ d, e \} \rangle \} \), so blue \( \succcurlyeq \)'s, red \( \succ \) hold for \( \succcurlyeq'_{\nu} \). *Also* deFi’s axioms hold of \( \succcurlyeq'_{\nu} \). ✓

Thus, we’ve finished the proof: we have a relation \( \succcurlyeq'_{\nu} \) that satisfies de Finetti’s axioms but is not probabilistically representable!
We have shown:

**Fact (Kraft, Pratt, and Seidenberg)**

There is a total preorder \( \succcurlyeq \) on \( \wp(\{a, b, c, d, e\}) \) such that:

1. \( \succcurlyeq \) is nontrivial: \( \emptyset \not\succcurlyeq W \);
2. \( \succcurlyeq \) is nonnegative: \( A \succcurlyeq \emptyset \);
3. \( \succcurlyeq \) is quasi-additive: if \( (A \cup B) \cap C = \emptyset \), then
   \[ A \succcurlyeq B \iff A \cup C \succcurlyeq B \cup C. \]
4. \( \succcurlyeq \) is *not probabilistically representable*. 
The KPS ordering is such that:

\[
\{a\} \succ \{b, c\} \quad \{c, d\} \succ \{a, b\} \quad \{b, e\} \succ \{a, c\} \quad \{a, b, c\} \succ \{d, e\}.
\]

For example:

1. Our guest is more likely to arrive on American than on British or Continental;
2. She is more likely to arrive on Continental or Delta than on American or British;
3. She is more likely to arrive on British or Emirates than on American or Continental;
4. She is more likely to arrive on American, British, or Continental than on Delta or Emirates.

Fine, *Theories of Probability*, p. 24: “what in our understanding of CP compels us to eliminate the ordering . . . as a CP relation?”
Outline

1. Basic Notions ✓
2. Considering Axioms: Rationality and Realism ✓
   • de Finetti’s Axioms ✓
3. Comparative Probability as Modal Logic ✓
4. Probabilistic Representability
   • The Kraft-Pratt-Seidenberg Counterexample ✓
   • Cancellation Axioms and Scott’s Representation Theorem
5. Segerberg and Gardenfors’ Completeness Theorem
6. Bonus 1: Imprecise Representability (by a set of measures)
7. Bonus 2: Extending an Ordering on a Set to the Powerset
We have shown:

**Fact (Kraft, Pratt, and Seidenberg)**

There is a total preorder $\succcurlyeq$ on $\mathcal{P}(\{a, b, c, d, e\})$ such that:

1. $\succcurlyeq$ is nontrivial: $\emptyset \not\succcurlyeq W$;
2. $\succcurlyeq$ is nonnegative: $A \succcurlyeq \emptyset$;
3. $\succcurlyeq$ is quasi-additive: if $(A \cup B) \cap C = \emptyset$, then $A \succcurlyeq B \iff A \cup C \succcurlyeq B \cup C$;
4. $\succcurlyeq$ is *not probabilistically representable*.

**QUESTION**: what further conditions on $\succcurlyeq$ must we assume for probabilistic representability?
Finite Cancellation

Two sequences $A_1, \ldots, A_n$ and $B_1, \ldots, B_n$ of subsets of $W$ are \textit{balanced} iff for all $w \in W$,

$$|\{i \mid w \in A_i\}| = |\{j \mid w \in B_j\}|.$$

Consider the \textbf{Finite Cancellation (FC) axiom}: for any balanced sequences $A_1, \ldots, A_n$ and $B_1, \ldots, B_n$ of subsets of $W$,

if $A_i \succsim B_i$ for all $i < n$, then $B_n \succsim A_n$. 
Let’s momentarily take both relations $\sim$ and $>$ as primitive.

Consider the axiom $\text{FC}_>$: for any balanced sequences $A_1, \ldots, A_n$ and $B_1, \ldots, B_n$ of subsets of $W$,

$$
\text{if } A_i \sim B_i \text{ for all } i < n, \text{ then } A_n \not> B_n.
$$

Fishburn’s “crude but instructive” **pragmatic motivation** for $\text{FC}_>$: an agent for whom $A_i \sim B_i$ for all $i < n$ and $A_n > B_n$ would presumably be willing to pay a positive amount to play this game:

- for each $A_i$ that obtains, he wins 1 Euro;
- for each $B_i$ that obtains, he loses 1 Euro.

Since $A_1, \ldots, A_n$ and $B_1, \ldots, B_n$ are balanced, no matter what is the true state of the world, his take from the game will be 0 Euros; but he paid a positive amount to play, so he’ll have a **sure loss**.
That pragmatic argument does not exactly work with \( FC \): for any balanced sequences \( A_1, \ldots, A_n \) and \( B_1, \ldots, B_n \) of subsets of \( W \),

if \( A_i \succ B_i \) for all \( i < n \), then \( B_n \succ A_n \).

An agent who violates this has \( A_i \succ B_i \) for all \( i < n \), and \( B_n \nless A_n \).

If we assume that \( B_n \nless A_n \) implies \( A_n > B_n \), then we can run the same argument. But if not, then can we claim that such an agent would be willing to pay some positive amount to play the game?
Two sequences $A_1, \ldots, A_n$ and $B_1, \ldots, B_n$ of subsets of $W$ are balanced iff for all $w \in W$,

$$|\{i \mid w \in A_i\}| = |\{j \mid w \in B_j\}|.$$ 

Consider the **Finite Cancellation (FC)** axiom: for any balanced sequences $A_1, \ldots, A_n$ and $B_1, \ldots, B_n$ of subsets of $W$,

if $A_i \succneq B_i$ for all $i < n$, then $B_n \succneq A_n$.

**Proposition (Finite Cancellation)**

If a relation $\succneq$ on $\wp(W)$ is probabilistically representable, then $\succneq$ satisfies **FC**.

**Proof.** On next slide...
Proposition (Finite Cancellation)

If a relation $\preceq$ on $\wp(W)$ is prob. representable, then for any balanced sequences $A_1, \ldots, A_n$ and $B_1, \ldots, B_n$ of subsets of $W$,$$
if A_i \preceq B_i \text{ for all } i < n, \text{ then } B_n \preceq A_n.
$$

Proof. We prove this for finite $W$, leaving the infinite case as an exercise. Let $\mu$ be the probability measure on $\wp(W)$ that agrees with $\preceq$. By the assumption that $A_1, \ldots, A_n$ and $B_1, \ldots, B_n$ are balanced, i.e., every $w \in W$ is in the same number of $A_i$’s and $B_i$’s, we have

$$
\sum_{i \leq n} \sum_{w \in A_i} \mu(\{w\}) = \sum_{i \leq n} \sum_{w \in B_i} \mu(\{w\}). \tag{1}
$$

Then by the additivity of $\mu$, (1) implies

$$
\sum_{i \leq n} \mu(A_i) = \sum_{i \leq n} \mu(B_i). \tag{2}
$$

Now if for all $i < n$, $A_i \preceq B_i$, so $\mu(A_i) \geq \mu(B_i)$, then (2) implies $\mu(B_n) \geq \mu(A_n)$, so $B_n \succeq A_n$. Thus, $\succeq$ satisfies the cancellation axioms.
Comparing Quasi-Additivity and Finite Cancellation

\[ FC_{\leq m} : \text{ for any } n \leq m \text{ and balanced sequences } A_1, \ldots, A_n \text{ and } B_1, \ldots, B_n \text{ of subsets of } W, \text{ if } A_i \succsim B_i \text{ for all } i < n, \text{ then } B_n \succsim A_n. \]

Fact
A relation \( \succsim \) on \( \mathcal{P}(W) \) satisfies \( FC_{\leq 3} \) iff \( \succsim \) is transitive and quasi-additive.

Proof. Let’s show \( FC_{\leq 3} \) implies *transitivity*. Since \( X, Y, Z \) and \( Y, Z, X \) are balanced, by \( FC_{\leq 3} \), \([X \succsim Y \text{ and } Y \succsim Z] \Rightarrow X \succsim Z.\)

To show that \( FC_{\leq 3} \) implies *quasi-additivity*, note that if
\( (A \cup B) \cap C = \emptyset \), then \( A, (B \cup C) \) and \( B, (A \cup C) \) are balanced. Thus, if \( A \succsim B \), then \( A \cup C \succsim B \cup C \). Moreover, \( (A \cup C), B \) and \( (B \cup C), A \) are balanced, if \( A \cup C \succsim B \cup C \), then \( A \succsim B \).
Comparing Quasi-Additivity and Finite Cancellation

**$FC_{\leq m}$**: for any $n \leq m$ and balanced sequences $A_1, \ldots, A_n$ and $B_1, \ldots, B_n$ of subsets of $W$, if $A_i \simeq B_i$ for all $i < n$, then $B_n \simeq A_n$.

**Fact**
A relation $\simeq$ on $\powerset(W)$ satisfies $FC_{\leq 3}$ iff $\simeq$ is transitive and quasi-additive.

**Proof.** Now let’s show that quasi-additivity implies $FC_{\leq 2}$. (The case of $FC_{\leq 3}$ is more difficult and left as an exercise.)

If $A_1, A_2$ and $B_1, B_2$ are balanced, then we have

\begin{align*}
A_1 - B_1 &= B_2 - A_2 \\
A_2 - B_2 &= B_1 - A_1.
\end{align*}

By quasi-additivity, if $A_1 \simeq B_1$, then $A_1 - B_1 \simeq B_1 - A_1$. So by the equations above, $B_2 - A_2 \simeq A_2 - B_2$, whence $B_2 \simeq A_2$. 

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KPS and Finite Cancellation

Recall that the KPS ordering had the following inequalities:

\[
\{a\} \succ \{b, c\} \quad \{c, d\} \succ \{a, b\} \quad \{b, e\} \succ \{a, c\} \quad \{a, b, c\} \succ \{d, e\}.
\]

This violates $\text{FC}_{\leq 4}$. The following sequences are balanced:

- $\langle A_1, A_2, A_3, A_4 \rangle$ be $\langle \{a\}, \{c, d\}, \{b, e\}, \{a, b, c\} \rangle$
- $\langle B_1, B_2, B_3, B_4 \rangle$ be $\langle \{b, c\}, \{a, b\}, \{a, c\}, \{d, e\} \rangle$.

Then in the KPS ordering, $A_i \succeq B_i$ for $i < 4$, but $B_4 \not\succeq A_4$. 
Representation Theorem

Theorem (Scott 1964)
For any finite set $W$, a binary relation $\succsim$ on $\mathcal{P}(W)$ is probabilistically representable iff it satisfies the nonnegativity, nontriviality, and finite cancellation axioms.

Proof. Where $|W| = n$, each $A \in \mathcal{P}(W)$ can be associated with a vector $\vec{A} \in \{0,1\}^n$, the “characteristic function” of $A$.

Let $\Gamma$ be the set of strict inequalities $A \succ B$, and $\Sigma$ the set of equivalences $A \sim B$. For $\gamma = A \succ B$, and $\sigma = A \sim B$, let

$$\vec{\gamma} = \vec{A} - \vec{B} \text{ and } \vec{\sigma} = \vec{A} - \vec{B}.$$
Proof. For $\gamma = A \succ B$, and $\sigma = A \sim B$, let

$$\overline{\gamma} = \overline{A} - \overline{B} \text{ and } \overline{\sigma} = \overline{A} - \overline{B}.$$ 

Lemma

There exists $c \in \mathbb{R}^n$ such that $c \cdot \overline{\gamma} > 0$ for all $\gamma \in \Gamma$, and $c \cdot \overline{\sigma} = 0$ for all $\sigma \in \Sigma$.

Given this lemma, we set:

$$\mu(A) = \frac{c \cdot \overline{A}}{c \cdot \overline{W}}.$$ 

Then:

- If $A \succ B$, then by the lemma, $\mu(A) > \mu(B)$;
- If $A \sim B$, then again by the lemma, $\mu(A) = \mu(B)$.
- Showing $\mu$ is a probability measure is straightforward.
Outline

1. Basic Notions ✓
2. Considering Axioms: Rationality and Realism ✓
   - de Finetti’s Axioms ✓
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4. Probabilistic Representability ✓
   - The Kraft-Pratt-Seidenberg Counterexample ✓
   - Cancellation Axioms, Scott’s Representation Theorem ✓
5. Segerberg and Gardenfors’ Completeness Theorem
6. Bonus 1: Imprecise Representability (by a set of measures)
7. Bonus 2: Extending an Ordering on a Set to the Powerset
Expressing \( FC \) in Our Formal Language

\( FC_m \): for any \( n \leq m \) and balanced sequences \( A_1, \ldots, A_n \) and \( B_1, \ldots, B_n \) of subsets of \( W \), if \( A_i \succsim B_i \) for all \( i < n \), then \( B_n \succsim A_n \).

Let’s now try to express the finite cancellation axioms in \( \mathcal{L}(\Box, \succeq) \).

Definition

Given a countable set \( At = \{p, q, r, \ldots\} \) of atomic sentences, the language \( \mathcal{L}(\Box, \succeq) \) is defined by the grammar:

\[
\varphi ::= p \mid \neg \varphi \mid (\varphi \land \varphi) \mid \Box \varphi \mid (\varphi \succeq \psi).
\]
Given a sequence \( \sigma = \langle \sigma_1, \ldots, \sigma_m \rangle \) with each \( \sigma_k \in \mathcal{L}(\supseteq) \), define for each \( i \leq n \) a formula \( N_i(\sigma) \) stating that \( |\{k \mid \sigma_k \text{ is true}\}| = i \) (note that this is not the same as \( |\{\sigma_k \mid \sigma_k \text{ is true}\}| = i \)):

\[
N_0(\sigma) = \neg \sigma_1 \land \cdots \land \neg \sigma_m;
\]

\[
N_1(\sigma) = (\sigma_1 \land \neg \sigma_2 \land \cdots \land \neg \sigma_n) \lor (\neg \sigma_1 \land \sigma_2 \land \neg \sigma_3 \land \cdots \land \neg \sigma_n) \cdots \lor (\neg \sigma_1 \land \cdots \land \neg \sigma_{m-1} \land \sigma_m).
\]

etc.

Next, given sequences \( \sigma \) and \( \sigma' \), both of length \( m \), define

\[
\sigma \mathcal{E} \sigma' := \bigvee_{0 \leq i \leq m} (N_i(\sigma) \land N_i(\sigma')).
\]

Finally, define \( \sigma \mathcal{E} \sigma' := \square(\sigma \mathcal{E} \sigma') \).

Thus, \( \sigma \mathcal{E} \sigma' \) says that every accessible world satisfies the same number of components from \( \sigma \) as from \( \sigma' \), as desired.
Expressing $FC$ in Our Formal Language

$FC$: for any balanced sequences $A_1, \ldots, A_n$ and $B_1, \ldots, B_n$ of subsets of $W$, if $A_i \precsim B_i$ for all $i < n$, then $B_n \precsim A_n$.

Now we can state finite cancellation as an infinite axiom schema:

$$FC \quad \phi_1 \ldots \phi_n \Im \psi_1 \ldots \psi_n \rightarrow (\bigwedge_{i<n} (\phi_i \succeq \psi_i)) \rightarrow (\psi_n \succeq \phi_n)$$
Segerberg and Gardenfors’ Completeness Theorem

System $FP$

**Taut** all tautologies

**Nec** \[
\frac{\phi}{\Box \phi}
\]

**MP** \[
\frac{\phi \rightarrow \psi}{\psi}
\]

**K** \[
\Box (\phi \rightarrow \psi) \rightarrow (\Box \phi \rightarrow \Box \psi)
\]

**Ex** \[
(\Box (\phi \leftrightarrow \phi') \land \Box (\psi \leftrightarrow \psi')) \rightarrow ((\phi \geq \psi) \leftrightarrow (\phi' \geq \psi'))
\]

**Bot** $\phi \geq \bot$

**BT** $\neg (\bot \geq \top)$

**Tot** $(\phi \geq \psi) \lor (\psi \geq \phi)$

**FC** \[
\phi_1 \ldots \phi_n \mathcal{E} \psi_1 \ldots \psi_n \rightarrow ((\bigwedge_{i<n} (\phi_i \geq \psi_i)) \rightarrow (\psi_n \geq \phi_n))
\]
Segerberg and Gardenfors’ Completeness Theorem

**System FA**

- **Taut** all tautologies
- **Nec** \( \frac{\varphi}{\square \varphi} \)
- **Ex** \( \frac{(\square (\varphi \leftrightarrow \varphi') \land \square (\psi \leftrightarrow \psi'))}{((\varphi \geq \psi) \leftrightarrow (\varphi' \geq \psi'))} \)
- **Bot** \( \varphi \geq \perp \)
- **BT** \( \neg (\perp \geq \top) \)
- **Tot** \( (\varphi \geq \psi) \lor (\psi \geq \varphi) \)
- **Tran** \( (\varphi \geq \psi) \rightarrow ((\psi \geq \chi) \rightarrow (\varphi \geq \chi)) \)
- **A** \( \neg \Diamond ((\varphi \lor \psi) \land \chi) \rightarrow (\varphi \geq \psi \leftrightarrow ((\varphi \lor \chi) \geq (\psi \lor \chi))) \)

- **MP** \( \frac{\varphi \rightarrow \psi}{\varphi \rightarrow \psi} \)
- **K** \( \square (\varphi \rightarrow \psi) \rightarrow (\square \varphi \rightarrow \square \psi) \)
Segerberg and Gardenfors' Completeness Theorem

**System FP**

- **Taut** all tautologies

- **Nec** \( \frac{\varphi}{\Box \varphi} \)

- **MP** \( \frac{\varphi \rightarrow \psi}{\psi} \)

- **K** \( \Box (\varphi \rightarrow \psi) \rightarrow (\Box \varphi \rightarrow \Box \psi) \)

- **Ex** \( (\Box (\varphi \leftrightarrow \varphi') \land \Box (\psi \leftrightarrow \psi')) \rightarrow ((\varphi \geq \psi) \leftrightarrow (\varphi' \geq \psi')) \)

- **Bot** \( \varphi \geq \bot \)

- **BT** \( \neg (\bot \geq \top) \)

- **Tot** \( (\varphi \geq \psi) \lor (\psi \geq \varphi) \)

- **FC** \( \varphi_1 \ldots \varphi_n \land \psi_1 \ldots \psi_n \rightarrow ((\land_{i<n} (\varphi_i \geq \psi_i)) \rightarrow (\psi_n \geq \varphi_n)) \)
Comparing \textbf{FA} and \textbf{FP}

Recall that earlier we showed:

\textbf{Fact}
A relation $\succeq$ on $\wp(W)$ satisfies $FC_{\leq 3}$ iff $\succeq$ is transitive and quasi-additive.

Similarly, where \textbf{FP}_3 system obtained by deleting from \textbf{FP} all instances of \textbf{FC} for $n > 3$, we have:

\textbf{Fact (van der Hoek)}
\textbf{FA} and \textbf{FP}_3 have the same theorems.
System **FP**

\[
\begin{align*}
\text{Ex} & \quad (\Box(\varphi \leftrightarrow \varphi') \land \Box(\psi \leftrightarrow \psi')) \rightarrow ((\varphi \geq \psi) \leftrightarrow (\varphi' \geq \psi')) \\
\text{Bot} & \quad \varphi \geq \bot \\
\text{BT} & \quad \neg(\bot \geq \top) \\
\text{Tot} & \quad (\varphi \geq \psi) \lor (\psi \geq \varphi) \\
\text{FC} & \quad \varphi_1 \ldots \varphi_n \EE \psi_1 \ldots \psi_n \rightarrow ((\bigwedge_{i<n} (\varphi_i \geq \psi_i)) \rightarrow (\psi_n \geq \varphi_n))
\end{align*}
\]

**Theorem (Segerberg 1971, Gärdenfors 1975)**

**FP** is sound and complete with respect to the class of models \( M = \langle W, R, \succsim, V \rangle \) for \( \mathcal{L}(\Box, \leq) \) such that for every \( w \in W \), \( \succsim_w \) is probabilistically representable.

**FP** also has the finite model property with respect to that class.
Definition (Probabilistic Kripke Model)

A **probabilistic Kripke model** is a tuple 
\( \mathcal{M} = \langle W, R, \{\mu_w\}_{w \in W}, V \rangle \) such that:

1. \( W \) is a nonempty set;

2. \( R \) is a **serial** binary relation on \( W \), i.e., such that for all \( w \in W \), \( R(w) = \{v \in W \mid wRv\} \neq \emptyset \);

3. \( \mu_w : \wp(R(w)) \rightarrow [0, 1] \) such that \( \mu_w(R(w)) = 1 \) and if \( A \cap B = \emptyset \), then \( \mu_w(A \cup B) = \mu_w(A) + \mu_w(B) \);

4. \( V : At \rightarrow \wp(W) \).

...
Semantics

Definition (Truth and Consequence)

Where $\mathcal{M}$ is a probabilistic Kripke model and $\varphi \in \mathcal{L}(\square, \triangleright)$, we define $\mathcal{M}, w \models \varphi$ ("$\varphi$ is true at $w$ in $\mathcal{M}$") by:

- $\mathcal{M}, w \models p$ iff $w \in V(p)$;
- $\mathcal{M}, w \models \neg \varphi$ iff $\mathcal{M}, w \not\models \varphi$;
- $\mathcal{M}, w \models (\varphi \land \psi)$ iff $\mathcal{M}, w \models \varphi$ and $\mathcal{M}, w \models \psi$;
- $\mathcal{M}, w \models \Box \varphi$ iff $\forall v \in R(w): \mathcal{M}, v \models \varphi$;
- $\mathcal{M}, w \models \varphi \triangleright \psi$ iff $\mu_w(\llbracket \varphi \rrbracket^\mathcal{M}_w) \geq \mu_w(\llbracket \psi \rrbracket^\mathcal{M}_w)$,

where $\llbracket \chi \rrbracket^\mathcal{M}_w = \{ v \in R(w) \mid \mathcal{M}, v \models \chi \}$.

The definition of consequence is standard.
Segerberg and Gardenfors’ Completeness Theorem

System FP

\[
\text{Ex} \ (\Box (\varphi \leftrightarrow \varphi') \land \Box (\psi \leftrightarrow \psi')) \rightarrow ((\varphi \geq \psi) \leftrightarrow (\varphi' \geq \psi'))
\]

\[
\text{Bot} \ \varphi \geq \bot
\]

\[
\text{BT} \ \neg (\bot \geq \top)
\]

\[
\text{Tot} \ (\varphi \geq \psi) \lor (\psi \geq \varphi)
\]

\[
\text{FC} \ \varphi_1 \ldots \varphi_n \models \psi_1 \ldots \psi_n \rightarrow ((\bigwedge_{i<n} (\varphi_i \geq \psi_i)) \rightarrow (\psi_n \geq \varphi_n))
\]

Theorem (Segerberg 1971, Gärdenfors 1975)

\textbf{FP} is sound and complete with respect to the class of probabilistic Kripke models for \(\mathcal{L}(\Box, \leq)\).

\textbf{FP} also has the finite model property with respect to that class.
Outline

1. Basic Notions ✓
2. Considering Axioms: Rationality and Realism ✓
   • de Finetti’s Axioms ✓
3. Comparative Probability as Modal Logic ✓
4. Probabilistic Representability ✓
   • The Kraft-Pratt-Seidenberg Counterexample ✓
   • Cancellation Axioms, Scott’s Representation Theorem ✓
5. Segerberg and Gardenfors’ Completeness Theorem ✓
6. Bonus 1: Imprecise Representability (by a set of measures)
7. Bonus 2: Extending an Ordering on a Set to the Powerset
Let $\mathcal{A}$ be an algebra on a nonempty set $W$, e.g., $\mathcal{A}$ is $\mathcal{P}(W)$.

**Definition (Agreement)**

Given a binary relation $\succeq$ on $\mathcal{A}$ and a set $\mathcal{P}$ of probability measures $\mu : \mathcal{A} \rightarrow [0, 1]$, $\mathcal{P}$ agrees with $\succeq$ iff for all $X, Y \in \mathcal{A}$:

$$X \succeq Y \iff \forall \mu \in \mathcal{P} : \mu(X) \geq \mu(Y).$$

We also say that $\mathcal{P}$ imprecisely represents $\succeq$.

**Definition (Representability)**

A binary relation $\succeq$ on $\mathcal{P}(W)$ is imprecisely representable iff it is imprecisely represented by a set of probability measures on $\mathcal{P}(W)$.

**QUESTION**: what axioms on $\succeq$ are necessary and sufficient for imprecise representability?
Generalized Finite Cancellation

Recall the Finite Cancellation (FC) axiom: for any balanced sequences \(A_1, \ldots, A_n\) and \(B_1, \ldots, B_n\) of subsets of \(W\),

\[
\text{if } A_i \gtrapprox B_i \text{ for all } i < n, \text{ then } B_n \gtrapprox A_n.
\]

Compare that to the Generalized Finite Cancellation (GFC) axiom:

if \(A_1, \ldots, A_{n-1}, \underbrace{A_n, \ldots, A_n}_{r \text{ times}}\) and \(B_1, \ldots, B_{n-1}, \underbrace{B_n, \ldots, B_n}_{r \text{ times}}\) \((r \geq 1)\)

are balanced sequences of subsets of \(W\), then

\[
\text{if } A_i \gtrapprox B_i \text{ for all } i < n, \text{ then } B_n \gtrapprox A_n.
\]
Representation Theorem

The Generalized Finite Cancellation (GFC) axiom:

if $A_1, \ldots, A_{n-1}, A_n, \ldots, A_n$ and $B_1, \ldots, B_{n-1}, B_n, \ldots, B_n$ ($r \geq 1$) are balanced sequences of subsets of $W$, then

if $A_i \simeq B_i$ for all $i < n$, then $B_n \simeq A_n$.

Theorem (Rios Insua 1992, Alon and Lehrer 2014)

For any finite set $W$, a binary relation $\simeq$ on $\wp(W)$ is imprecisely representable iff it satisfies reflexivity, nontriviality, nonnegativity, and GFC.
System FG

Taut  all tautologies

Nec  $\frac{\varphi}{\Box \varphi}$

MP  $\frac{\varphi \rightarrow \psi}{\psi}$

K  $\Box (\varphi \rightarrow \psi) \rightarrow (\Box \varphi \rightarrow \Box \psi)$

Ex  $(\Box (\varphi \leftrightarrow \varphi') \land \Box (\psi \leftrightarrow \psi')) \rightarrow ((\varphi \geq \psi) \leftrightarrow (\varphi' \geq \psi'))$

Bot  $\varphi \geq \bot$

BT  $\neg (\bot \geq \top)$

GFC  $\varphi_1 \ldots \varphi_{n-1} \varphi_n \ldots \varphi_n \underbrace{\mathbb{E} \psi_1 \ldots \psi_{n-1}}_{r \text{ times}} \psi_n \ldots \psi_n \rightarrow$

$$((\land_{i<n} (\varphi_i \geq \psi_i)) \rightarrow (\psi_n \geq \varphi_n))$$
Soundness and Completeness

\[
\begin{align*}
\text{Bot} & \quad \varphi \geq \perp \\
\text{BT} & \quad \neg (\perp \geq \top) \\
\text{GFC} & \quad \varphi_1 \ldots \varphi_{n-1} \varphi_n \ldots \varphi_n \mathbb{E} \psi_1 \ldots \psi_{n-1} \psi_n \ldots \psi_n \rightarrow \\
& \quad \left( (\bigwedge_{i<n} (\varphi_i \geq \psi_i)) \rightarrow (\psi_n \geq \varphi_n) \right)
\end{align*}
\]

**Theorem (Alon and Heifetz 2014)**

The system \( \textbf{FG} \) is sound and complete with respect to the class of QP Kripke models in which every \( \preceq_w \) is imprecisely representable.
Sets-of-Measures Kripke Models

Definition (Sets-of-Measures Kripke Model) A 

\textit{sets-of-measures Kripke model} is a tuple 

\[ M = \langle W, R, \{P_w\}_{w \in W}, V \rangle \]

such that:

1. \( W \) is a nonempty set;

2. \( R \) is a \textit{serial} binary relation on \( W \), i.e., such that for all \( w \in W \), \( R(w) = \{v \in W \mid wRv\} \neq \emptyset \);

3. \( P_w \) is a set of functions \( \mu_w : \wp(R(w)) \rightarrow [0, 1] \) such that 
   \( \mu_w(R(w)) = 1 \) and if \( A \cap B = \emptyset \), then 
   \( \mu_w(A \cup B) = \mu_w(A) + \mu_w(B) \);

4. \( V : \text{At} \rightarrow \wp(W) \).
Definition (Truth and Consequence)

Where $\mathcal{M}$ is a set-of-measures Kripke model and $\varphi \in \mathcal{L}(\square, \geq)$, we define $\mathcal{M}, w \models \varphi$ by (with other clauses as before):

$\Rightarrow \mathcal{M}, w \models \varphi \geq \psi$ iff for all $\mu_w \in \mathcal{P}_w$: $\mu_w([\varphi]^\mathcal{M}_w) \geq \mu_w([\psi]^\mathcal{M}_w)$,

where $[\chi]^\mathcal{M}_w = \{v \in R(w) \mid \mathcal{M}, v \models \chi\}$.

The definition of consequence is standard.
Soundness and Completeness

Bot  \( \varphi \geq \bot \)

BT  \( \neg (\bot \geq \top) \)

GFC  \( \varphi_1 \cdots \varphi_{n-1} \varphi_n \cdots \varphi_n \text{I}E\psi_1 \cdots \psi_{n-1} \psi_n \cdots \psi_n \rightarrow \)

\[ r \text{ times} \quad r \text{ times} \]

\[ (((\bigwedge_{i<n} (\varphi_i \geq \psi_i))) \rightarrow (\psi_n \geq \varphi_n)) \]

**Theorem (Alon and Heifetz 2014)**

The system **FG** is sound and complete with respect to the class of sets-of-measures Kripke models.
Bonuses

1. Basic Notions ✓
2. Considering Axioms: Rationality and Realism ✓
   • de Finetti’s Axioms ✓
3. Comparative Probability as Modal Logic ✓
4. Probabilistic Representability ✓
   • The Kraft-Pratt-Seidenberg Counterexample ✓
   • Cancellation Axioms, Scott’s Representation Theorem ✓
5. Segerberg and Gardenfors’ Completeness Theorem ✓
6. Bonus 1: Imprecise Representability ✓
7. Bonus 2: Extending an Ordering on a Set to the Powerset
Consider the English locution ‘at least as likely as’, as in

(1) It is at least as likely that our visitor is coming in on American Airlines as it is that he is coming on Continental Airlines.

What does this mean? Specifically, what is its logic?

Some entailments are clear. For instance, (1) follows from (2):

(2) American is at least as likely as Continental or Delta.

What else? How might we interpret such talk model-theoretically?
What is the relation between ordinary talk using ‘probably’ and ‘at least as likely as’ and the mathematical theory of probability?

Is Kolmogorovian probability implicated in their semantics?

Hamblin (1959, 234): “Metrical probability theory is well-established, scientifically important and, in essentials, beyond logical reproof. But when, for example, we say ‘It’s probably going to rain’, or ‘I shall probably be in the library this afternoon’, are we, even vaguely, using the metrical probability concept?”

Kratzer (2012, 25): “Our semantic knowledge alone does not give us the precise quantitative notions of probability and desirability that mathematicians and scientists work with.”
Definition (World-Ordering Model)

A (total) world-ordering model is a tuple \( M = \langle W, R, \{ \succeq_w | w \in W \}, V \rangle \):

- \( W \) is a non-empty set;
- \( R \) is a (serial) binary relation on \( W \); \( R(w) = \{ v \in W \mid wRv \} \);
- For each \( w \in W \), \( \succeq_w \) is a (total) preorder on \( R(w) \);
- \( V : \text{At} \to \wp(W) \) is a valuation function.

Following Lewis, we can lift \( \succeq_w \) to a relation \( \succeq^l_w \) on \( \wp(W) \):

\[
A \succeq^l_w B \iff \forall b \in B_w \exists a \in A_w : a \succeq_w b,
\]

where \( X_w = X \cap R(w) \).
Following Lewis, we can lift \( \succeq_w \) to a relation \( \succeq^l_w \) on \( \wp(W) \):

\[
A \succeq^l_w B \quad \text{iff} \quad \forall b \in B_w \exists a \in A_w : a \succeq_w b.
\]

**Definition (Truth)**

Given a world-ordering pointed model \( M, w \) and \( \varphi \in L(\Box, \succeq) \), we define \( M, w \vDash \varphi \) and \( \llbracket \varphi \rrbracket^M = \{ v \in W \mid M, v \vDash \varphi \} \) as follows:

\[
\begin{align*}
M, w \vDash p & \quad \text{iff} \quad w \in V(p); \\
M, w \vDash \neg \varphi & \quad \text{iff} \quad M, w \not\vDash \varphi; \\
M, w \vDash \varphi \land \psi & \quad \text{iff} \quad M, w \vDash \varphi \text{ and } M, w \vDash \psi; \\
M, w \vDash \Box \varphi & \quad \text{iff} \quad \forall v \in R(w) : M, v \vDash \varphi; \\
M, w \vDash \varphi \models \psi & \quad \text{iff} \quad \llbracket \varphi \rrbracket^M \succeq^l_w \llbracket \psi \rrbracket^M.
\end{align*}
\]
As pointed out by Yalcin (2010) and Lassiter (2010), Kratzer’s approach validates some rather dubious patterns. For instance, it predicts that (3) should follow from (1) and (2):

(1) **American** is at least as likely as **Continental**.

(2) **American** is at least as likely as **Delta**.

(3) **American** is at least as likely as **Continental or Delta**.

It also fails to validate some intuitively obvious patterns.
Kratzer’s Semantics

Definition (World-Ordering Model)
A (total) world-ordering model $\mathcal{M} = \langle \mathcal{W}, R, \{ \succeq_w \mid w \in \mathcal{W} \}, V \rangle$ has for each $w \in \mathcal{W}$ a (total) preorder $\succeq_w$ on $R(w)$.

Following Lewis, we can lift $\succeq_w$ to a relation $\succeq^l_w$ on $\mathcal{P}(\mathcal{W})$:

$$A \succeq^l_w B \iff \forall b \in B_w \exists a \in A_w : a \succeq_w b.$$ 

Kratzer gives the truth clause for $\succeq$ using the lifted relation $\succeq^l_w$.

Definition (Truth)
Given a pointed world-ordering model $\mathcal{M}$, $w$ and formula $\varphi$, we define $\mathcal{M}, w \models_l \varphi$ as follows (with the other clauses as before):

$$\mathcal{M}, w \models_l \varphi \succeq \psi \iff [\varphi]^M \succeq^l_w [\psi]^M.$$
Yalcin’s List of Intuitively Valid and Invalid Patterns

V1  $\triangle \varphi \rightarrow \neg \triangle \neg \varphi$

V2  $\triangle (\varphi \land \psi) \rightarrow (\triangle \varphi \land \triangle \psi)$

V3  $\triangle \varphi \rightarrow \triangle (\varphi \lor \psi)$

V4  $\varphi \geq \bot$

V5  $\top \geq \varphi$

V6  $\Box \varphi \rightarrow \triangle \varphi$

V7  $\triangle \varphi \rightarrow \Diamond \varphi$

V11  $(\psi \geq \varphi) \rightarrow (\triangle \varphi \rightarrow \triangle \psi)$

V12  $(\psi \geq \varphi) \rightarrow ((\varphi \geq \neg \varphi) \rightarrow (\psi \geq \neg \psi))$

I1  $((\varphi \geq \psi) \land (\varphi \geq \chi)) \rightarrow (\varphi \geq (\psi \lor \chi))$

I2  $(\varphi \geq \neg \varphi) \rightarrow (\varphi \geq \psi)$

I3  $\triangle \varphi \rightarrow (\varphi \geq \psi)$

E1  $(\triangle \varphi \land \triangle \psi) \rightarrow \triangle (\varphi \land \psi)$
Fact
V1-V10 and V12 are all valid over world-ordering models according to Kratzer’s semantics; V11 is not valid; l1-13 are all valid. X
Systems $W$, $F$, $WJ$, and $FJ$

System $W$ is the minimal modal logic $K$ plus:

Ex $\Box (\phi \leftrightarrow \phi') \land \Box (\psi \leftrightarrow \psi') \rightarrow ((\phi \geq \psi) \leftrightarrow (\phi' \geq \psi'))$

Bot $\phi \geq \bot$  BT $\neg (\bot \geq \top)$

Tran $(\phi \geq \psi) \rightarrow ((\psi \geq \chi) \rightarrow (\phi \geq \chi))$

Mon $\Box (\phi \rightarrow \psi) \rightarrow (\phi \geq \psi)$

System $F$ is $W$ plus:

Tot $(\phi \geq \psi) \lor (\psi \geq \phi)$

System $WJ$ (resp. $FJ$) is $W$ (resp. $F$) plus:

J $((\phi \geq \psi) \land (\phi \geq \chi)) \rightarrow (\phi \geq (\psi \lor \chi))$
Kratzer’s Semantics

Theorem (Axiomatization of Kratzer’s Semantics)

1. \textbf{WJ} is sound and complete with respect to the class of world-ordering models with \textit{Lewis’s lifting}.
2. \textbf{FJ} is sound and complete with respect to the class of \textit{total} world-ordering models with \textit{Lewis’s lifting}. 

Wes Holliday and Thomas Icard: Comparative Probability
Kratzer’s Semantics

Definition (World-Ordering Model)
A (total) world-ordering model $\mathcal{M} = \langle W, R, \{\succeq_w \mid w \in W \}, V \rangle$ has for each $w \in W$ a (total) preorder $\succeq_w$ on $R(w)$.

Following Lewis, we can lift $\succeq_w$ to a relation $\succeq^l_w$ on $\mathcal{P}(W)$:

$$A \succeq^l_w B \text{ iff } \forall b \in B \exists a \in A \colon a \succeq_w b.$$  

Kratzer gives the truth clause for $\geq$ using the lifted relation $\succeq^l_w$.

Definition (Truth)
Given a pointed world-ordering model $\mathcal{M}$, $w$ and formula $\phi$, we define $\mathcal{M}, w \models_I \phi$ as follows (with the other clauses as before):

$$\mathcal{M}, w \models_I \phi \geq \psi \text{ if and only if } \llbracket \phi \rrbracket^\mathcal{M} \succeq^l_w \llbracket \psi \rrbracket^\mathcal{M}.$$
Kratzer’s Semantics

Definition (World-Ordering Model)
A (total) world-ordering model $\mathbf{M} = \langle W, R, \{\succeq_w \mid w \in W \}, V \rangle$
has for each $w \in W$ a (total) preorder $\succeq_w$ on $R(w)$.

Following Lewis, we can lift $\succeq_w$ to a relation $\succeq^l_w$ on $\mathcal{P}(W)$:

\[ A \succeq^l_w B \iff \exists \text{ function } f : B_w \to A_w \text{ s.th. } \forall x \in B_w : f(x) \succeq_w x. \]

Kratzer gives the truth clause for $\geq$ using the lifted relation $\succeq^l_w$.

Definition (Truth)
Given a pointed world-ordering model $\mathbf{M}$, $w$ and formula $\varphi$, we define $\mathbf{M}, w \Vdash \varphi$ as follows (with the other clauses as before):

\[ \mathbf{M}, w \Vdash \varphi \geq \psi \iff [\varphi]^{\mathbf{M}} \succeq^l_w [\psi]^{\mathbf{M}}. \]
Bonus 2: Extending an Ordering on a Set to the Powerset

An Alternative Lifting

Definition (World-Ordering Model)

A (total) world-ordering model $\mathcal{M} = \langle W, R, \{\succeq_w \mid w \in W\}, V \rangle$ has for each $w \in W$ a (total) preorder $\succeq_w$ on $R(w)$.

Here is another way to lift $\succeq_w$ to a relation $\succeq^w_\uparrow$ on $\wp(W)$:

$A \succeq^w_\uparrow B \iff \exists$ injection $f : B_w \to A_w$ s.th. $\forall x \in B_w : f(x) \succeq_w x$.

Definition (Truth)

Given a pointed world-ordering model $\mathcal{M}$, $w$ and formula $\varphi$, we define $\mathcal{M}, w \models^\uparrow \varphi$ as follows (with the other clauses as before):

$\mathcal{M}, w \models^\uparrow \varphi \geq \psi \iff \llbracket \varphi \rrbracket^\mathcal{M} \succeq^w_\uparrow \llbracket \psi \rrbracket^\mathcal{M}$. 
An Alternatively Lifting

Given $a \succ b \succ c \succ d$, consider the liftings:

\[
\begin{align*}
abcd & \succeq^l abcd & \succeq^l abc & \succeq^l abd & \succeq^l acd & \succeq^l ab & \succeq^l ac & \succeq^l ad & \succeq^l a & \succ^l b & \succ^l c & \succ^l d & \succ^l \emptyset
\end{align*}
\]

\[
\begin{align*}
abcd & \succ^\uparrow abcd & \succ^\uparrow abc & \succ^\uparrow abd & \succ^\uparrow acd & \succ^\uparrow bcd & \succ^\uparrow ab & \succ^\uparrow ac & \succ^\uparrow bc & \succ^\uparrow ad & \succ^\uparrow bd & \succ^\uparrow cd & \succ^\uparrow a & \succ^\uparrow b & \succ^\uparrow c & \succ^\uparrow d & \succ^\uparrow \emptyset
\end{align*}
\]

**Figure:** Comparison of Lewis's lifting $\succeq^l$ and the new lifting $\succ^\uparrow$
System \textbf{FG}

\textbf{Taut} all tautologies

\textbf{Nec} \quad \frac{\phi}{\square \phi}

\textbf{MP} \quad \frac{\phi \rightarrow \psi}{\psi}

\textbf{K} \quad \square (\phi \rightarrow \psi) \rightarrow (\square \phi \rightarrow \square \psi)

\textbf{Ex} \quad (\square (\phi \leftrightarrow \phi') \land \square (\psi \leftrightarrow \psi')) \rightarrow ((\phi \geq \psi) \leftrightarrow (\phi' \geq \psi'))

\textbf{Bot} \quad \phi \geq \bot

\textbf{BT} \quad \neg (\bot \geq \top)

\textbf{GFC} \quad \phi_1 \ldots \phi_{n-1} \phi_n \ldots \phi_n \underbrace{\bigwedge_{i<n} (\phi_i \geq \psi_i)}_{r \text{ times}} \rightarrow \underbrace{\bigwedge_{i<n} (\psi_i \geq \psi_n)}_{r \text{ times}} \rightarrow (\psi_n \geq \phi_n)
Soundness and Completeness

Here is a better way to lift $\succeq_w$ to a relation $\succeq_w^\uparrow$ on $\mathcal{P}(W)$:

$$ A \succeq_w^\uparrow B \text{ iff } \exists \text{ injective } f : B \to A \text{ s.th. } \forall x \in B : f(x) \succeq_w x $$

**Proposition (Harrison-Trainor, Holliday, and Icard)**

$\text{FG}$ is sound and complete with respect to the class of path-finite\(^1\) world-ordering models with the $\uparrow$ lifting.

**Moral:** simply changing Kratzer’s semantics by requiring that the function be injective yields the same logic of ‘at least as likely as’ as a semantics based on sets of probability measures.

---

\(^1\)I.e., there is no infinite path $x_1 \preceq_w x_2 \preceq_w x_3 \ldots$ with $x_i \neq x_j$ for $i \neq j$. 

Wes Holliday and Thomas Icard: Comparative Probability
Soundness and Completeness

Here is a better way to lift $\preceq_w$ to a relation $\preceq_{w}^{\uparrow}$ on $\wp(W)$:

$$A \preceq_{w}^{\uparrow} B \text{ iff } \exists \text{ injective } f : B \to A \text{ s.th. } \forall x \in B : f(x) \succeq x$$

**Proposition (Harrison-Trainor, Holliday, and Icard)**

**FG** is sound and complete with respect to the class of path-finite\(^2\) world-ordering models with the $\uparrow$ lifting.

**Moral:** all of Yalcin’s intuitively validities are validated; none of his intuitive invalidities are validated; thus, the semantics avoids the entailment problems raised for Kratzer’s semantics.

\(^2\)I.e., there is no infinite path $x_1 \preceq_w x_2 \preceq_w x_3 \ldots$ with $x_i \neq x_j$ for $i \neq j$. 
Outline

1. Basic Notions ✓
2. Considering Axioms: Rationality and Realism ✓
   - de Finetti's Axioms ✓
3. Comparative Probability as Modal Logic ✓
4. Probabilistic Representability ✓
   - The Kraft-Pratt-Seidenberg Counterexample ✓
   - Cancellation Axioms, Scott’s Representation Theorem ✓
5. Segerberg and Gardenfors’ Completeness Theorem ✓
6. Bonus 1: Imprecise Representability ✓
7. Bonus 2: Extending an Ordering on a Set to the Powerset ✓
Thank you!