1 How much do you know?

1. Translate the following sentences into FOL twice, using universal quantifiers for one translation and existential quantifiers for the other.

   (a) Every cube is left of every tetrahedron.
   \[ \forall x \forall y (\text{Cube}(x) \land \text{Tet}(y) \rightarrow \text{LeftOf}(x,y)) \]

   (b) Some cube is left of some tetrahedron.
   \[ \exists x \exists y (\text{Cube}(x) \land \text{Tet}(y) \land \text{LeftOf}(x,y)) \]

   (c) No cube is left of any tetrahedron.
   \[ \forall x \forall y ((\text{Cube}(x) \land \text{Tet}(y)) \rightarrow \neg \text{LeftOf}(x,y)) \]

   (d) Some cube is not left of some tetrahedron.
   The following might be okay, especially given that this part of the course was dealing with multiple quantifiers of the same type:
   \[ \exists x \exists y (\text{Cube}(x) \land \text{Tet}(y) \land \neg \text{LeftOf}(x,y)) \]
   But really the sentence is saying that there is a cube which is not left of any tetrahedron. Translate that as follows:
   \[ \exists x (\text{Cube}(x) \land \forall y (\text{Tet}(y) \rightarrow \neg \text{LeftOf}(x,y))) \]

2. Carefully translate the following sentences.

   (a) There is a pair of small cubes.
   \[ \exists x \exists y (x \neq y \land \text{Cube}(x) \land \text{Small}(x) \land \text{Cube}(y) \land \text{Small}(y)) \]

   (b) There is a cube to the left of some cube.
   \[ \exists x \exists y (\text{Cube}(x) \land \text{Cube}(y) \land \text{LeftOf}(x,y)) \]

   (c) For every pair of tetrahedra, one is smaller than the other.
   \[ \forall x \forall y ((\text{Tet}(x) \land \text{Tet}(y) \land x \neq y) \rightarrow (\text{Smaller}(x,y) \lor \text{Larger}(x,y))) \]
(d) There is a pair of cubes to the left of some cube.

\[ \exists x \exists y \exists z (\text{Cube}(x) \land \text{Cube}(y) \land x \neq y \land \text{Cube}(z) \land \text{LeftOf}(x,z) \land \text{LeftOf}(y,z)) \]

(e) Every tetrahedron is the same size.

\[ \forall x (\text{Tet}(x) \rightarrow \text{Small}(x)) \lor \forall x (\text{Tet}(x) \rightarrow \text{Medium}(x)) \lor \forall x (\text{Tet}(x) \rightarrow \text{Large}(x)) \]

But you could also say:

\[ \forall x \forall y ((\text{Tet}(x) \land \text{Tet}(y)) \rightarrow \text{SameSize}(x,y)) \]

3. (Ex 11.7) Suppose a column containing blocks is called a party (think of the blocks as those attending the party). A party is lonely if there’s only one block attending it. A party is exclusive if there’s some block which is not attending it (that is, it is in another column).

(a) Translate the following eight sentences into sensible English.

i. \[ \exists x \exists y (x \neq y \land \text{SameCol}(x,y)) \]
   There’s a party that isn’t lonely!

ii. \[ \exists x \exists y (x \neq y \land \neg \text{SameCol}(x,y)) \]
   There’s more than one exclusive party!

iii. \[ \exists x \neg \exists y (x \neq y \land \text{SameCol}(x,y)) \]
   There’s a party that’s lonely.

iv. \[ \neg \exists x \exists y (x \neq y \land \text{SameCol}(x,y)) \]
   There’s no party that isn’t lonely.

v. \[ \forall x \forall y (x \neq y \rightarrow \neg \text{SameCol}(x,y)) \]
   There’s no more than one party. (Since we always assume the world is non-empty, you could say: there’s exactly one party)

vi. \[ \forall x \forall y (x \neq y \rightarrow \neg \text{SameCol}(x,y)) \]
   There’s no party that isn’t lonely.

vii. \[ \forall x \neg \forall y (x \neq y \rightarrow \text{SameCol}(x,y)) \]
   Every party is exclusive!

viii. \[ \neg \forall x \forall y (x \neq y \rightarrow \text{SameCol}(x,y)) \]
   There’s more than one exclusive party!

(b) Which of the sentences above are equivalent to each other? Which are negations of each other?

(ii) and (viii) are equivalent.

(iv) and (vi) are equivalent.

(i) and (iv) are negations of each other.

(v) and (viii) are negations of each other.
4. (Ex 11.11) Create a world (as if in Tarski's World, no larger than 8 by 8) in which the following ten sentences are all true.

Note: It's really helpful to start off with some colloquial translations before trying to solve this kind of puzzle. So I'll do that.

(a) $\exists x \exists y \exists z (\text{Cube}(x) \land \text{Dodec}(y) \land \text{Tet}(z))$

There's (at least) a cube, a dodecahedron and a tetrahedron.

(b) $\neg \exists x \text{Large}(x)$

Nothing is large.

(c) $\forall x (\text{Dodec}(x) \rightarrow \exists y (\text{Cube}(y) \land \text{BackOf}(x,y)))$

Every dodecahedron has some cube in front of it.

(d) $\forall x (\text{Tet}(x) \rightarrow \exists y \exists z (\text{Between}(x,y,z)))$

Every tetrahedron is between two blocks.

(e) $\forall x \forall y \forall z (\text{Between}(x,y,z) \rightarrow \text{Larger}(x,y))$

Every block is larger than one of the blocks it lies between.

(f) $\exists x \exists y (x \neq y \land \forall w ((w=x \lor w=y) \rightarrow \forall z \neg \text{BackOf}(z,w)))$

There is a pair of blocks, each of which has nothing behind it.

(g) $\forall x (\text{Cube}(x) \leftrightarrow \exists y (\text{Tet}(y) \land \text{BackOf}(y,x)))$

Each cube – and only each cube – has a tetrahedron behind it.

(h) $\forall x \forall y (\text{Larger}(x,y) \rightarrow \exists z \text{Between}(x,y,z))$

Whenever some block is larger than another block, the larger block is between the smaller block and a third block.

(i) $\neg \forall x \forall y (\text{LeftOf}(x,y) \lor \text{RightOf}(x,y))$

It's not the case that every block is left or right of another block. (But this could be true if there's at least one block, since it is neither left of nor right of itself.)

(j) $\exists x \exists y (\neg (\text{FrontOf}(x,y) \lor \text{BackOf}(x,y)))$

It's not the case that every block is in front of or behind another block. (Again, this is true whenever there is some block, since it is neither in front of nor behind itself.)

The following world works:
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