Beyond Worlds and Accessibility

Part 1

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EASSLC 2014, July 2-8
Course Description

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In each case, we will study the logical properties of these classes of models, illuminating the philosophical issues at stake.
Outline of the Course

- Day 1 - Introduction and Background
- Day 1-3 - RA and Subjunctivist Semantics
- Day 3 - Set-Selection Function Semantics
- Day 4 - Possibility Semantics
- Day 5 - Situation Semantics
For the slides and references for this course, go to

http://philosophy.berkeley.edu/holliday

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Recent work on related topics comes from researchers in China! See, e.g., papers by Zhaoqing Xu and Chenwei Shi on the webpage given above.
From my “Summary of “Epistemic Closure and Epistemic Logic I: Relevant Alternatives and Subjunctivism”,” Logic Across the University:

An application of formal methods in philosophy gains value insofar as:

(1) it is faithful to the philosophical views being formalized;
(2) it can handle concrete examples discussed in the philosophical literature;
(3) it goes beyond particular examples to provide a systematic and general view of the topic;
(4) it leads to philosophically-relevant discoveries that would be difficult to make by non-formal methods alone;
(5) it leads to the development of new views that solve previous philosophical problems.

(1) and (2) address the worry that by formalizing we may “change the subject.” (3) - (5) address the question, “What do we get out of this?” Of course, there are other features that contribute to the value of an application of formal methods, as well as features that detract from it.
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Motivation for Today

Basic Epistemic Logic & What One Knows

Model what an agent knows by what is true throughout a set $R^a(w)$ of possibilities that are compatible with her knowledge in world $w$:

$$M, v \models K^a_j, \forall v^2 \in R^a(w): M, v \models j.$$

Q: What exactly do $v$ have to be "compatible"?

A: $v^2 \in R^a(w)$ means that everything $a$ knows in $w$ is true in $v$.

Q: So you're explaining when '$K^a_j$' is true in terms of knowledge itself?

A: Yes.

Q: Isn't that a problem? Somehow cheating or conceptually circular?

A: No! The point of basic epistemic models is to represent the content of an agent's knowledge—not to give a substantive account of what it takes to know something in other terms.
Basic Epistemic Logic & What One Knows

Model what an agent $a$ knows by what is true throughout a set $R_a(w)$ of possibilities that are compatible with her knowledge in world $w$:

$$M, w \models K_a \varphi \iff \forall v \in R_a(w) : M, v \models \varphi.$$  \hspace{1cm} (1)
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Q: what exactly does it mean for $\nu$ to be “compatible”?

A: $\nu \in R_a(w)$ means that everything $a$ knows in $w$ is true in $\nu$.

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Model what an agent knows by what is true throughout a set $R(w)$ of possibilities that are “compatible” with her knowledge in world $w$:

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Qualification: two aspects of these models are not neutral with respect to different substantive accounts of what it takes to know something.
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Most obviously, if we impose conditions on $R_a$ beyond reflexivity [$w \in R_a(w)$] such as symmetry [$v \in R_a(w) \Rightarrow w \in R_a(v)$], we build in substantive views (in this case, that $\neg \varphi \rightarrow K_a \neg K_a \varphi$ should be valid) that conflict with some accounts of knowledge (ditto for ‘information’).
Motivation for Today

Basic Epistemic Logic & What One Knows

\[ \mathcal{M}, w \models K_a \varphi \iff \forall v \in R_a(w) : \mathcal{M}, v \models \varphi. \]  

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Less obviously, while the $\Rightarrow$ direction of (1) follows immediately from the stipulation that $v \in R_a(w)$ means everything $a$ knows in $w$ is true in $v$, the $\Leftarrow$ direction of (1) does not follow just from that stipulation.
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In fact, to add the \( \Leftarrow \) direction is just to assume the principle that knowledge is closed under known implication:

\[ (K_a \varphi \land K_a(\varphi \rightarrow \psi)) \rightarrow K_a \psi \]
Basic Epistemic Logic & What One Knows

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Basic Epistemic Logic & What One Knows

\[ M, w \models K_a \varphi \iff \forall \nu \in R_a(w) : M, \nu \models \varphi. \]  

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The \( \Leftarrow \) direction conflicts with some accounts of knowledge I will discuss.
A: The point of basic epistemic models is to represent the content of an agent’s knowledge—what the agent knows—not to give a substantive account of what it takes to know something in other terms.

Those who criticize basic epistemic models for not giving more substantive insight into what it takes to know something about the world (or what it takes to get information from the world) are missing the point.
Epistemic Logic & What One Knows

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The beauty of it is that basic epistemic logic can nevertheless give illuminating analyses of a wide range of epistemic phenomena, especially in *multi-agent* settings with information change (think Muddy Children).
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For many examples, see:

Motivation for Today

Epistemic Accessibility vs. Indistinguishability

Distinction: accessibility vs. other notions of **indistinguishability**.
Motivation for Today

Epistemic Accessibility vs. Indistinguishability

Distinction: accessibility vs. other notions of indistinguishability.

Suppose that we replace $R$ by a binary relation $E$ on $W$, where our intuitive interpretation is that $wEv$ holds “iff the subject’s perceptual experience and memory” in scenario $v$ “exactly match his perceptual experience and memory” in scenario $w$ (Lewis, “Elusive Knowledge”).
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Epistemic Accessibility vs. Indistinguishability

Distinction: accessibility vs. other notions of **indistinguishability**.

Suppose that we replace $R$ by a binary relation $E$ on $W$, where our intuitive interpretation is that $wEv$ holds “iff the subject’s perceptual experience and memory” in scenario $v$ “exactly match his perceptual experience and memory” in scenario $w$ (Lewis, “Elusive Knowledge”).

$$\mathcal{M}, w \models K_a \varphi \iff \forall v \in E_a(w) : \mathcal{M}, v \models \varphi.$$

Hence the agent knows $\varphi$ in $w$ just in case $\varphi$ is true in all scenarios that are experientially indistinguishable from $w$ for the agent.
Epistemic Accessibility & Indistinguishability

Important differences between the picture with $E$ and the one with $R$:
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- The epistemic model with $E$ does not simply represent the content of one’s knowledge; rather, it commits us to a particular view of the conditions under which an agent has knowledge, specified in terms of perceptual experience and memory.
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Epistemic Accessibility & Indistinguishability

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- The epistemic model with $E$ does not simply represent the content of one’s knowledge; rather, it commits us to a particular view of the conditions under which an agent has knowledge, specified in terms of perceptual experience and memory.

- It is plausible that $E$ has certain properties, such as symmetry ($wEv$ iff $vEw$), which are questionable as properties of $R$. 
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- It is plausible that $E$ has certain properties, such as symmetry ($wE v$ iff $vE w$), which are questionable as properties of $R$.

Since the properties of the relation will determine valid principles for the knowledge operator $K$, we must be clear about which interpretation of the relation we adopt.
“Considering Possible”

\[ M, w \models K_a \varphi \iff \forall v \in R_a(w) : M, v \models \varphi. \]  

(1)

While one may read \( wRv \) as

- **“for all the agent knows** in \( w \), scenario \( v \) is the scenario she is in,”

one should **not** read \( wRv \) as

- **“in \( w \), the agent considers scenario \( v \) possible,”**

where the latter suggest a subjective psychological notion.
“Consideration Possible”

\[ M, w \models K_a \phi \iff \forall v \in R_a(w) : M, v \models \phi. \] (1)

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An agent may not subjectively consider it possible that his friend, whom he has regarded for years as his most trusted ally, has betrayed him.
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one should not read \( wRv \) as

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where the latter suggest a subjective psychological notion.

An agent may not subjectively consider it possible that his friend, whom he has regarded for years as his most trusted ally, has betrayed him. It obviously does not follow that he knows that his friend has not betrayed him, as it would according to the subjective reading of \( R \) with (1).
Motivation for Today

Basic Epistemic Logic & What One Knows

Model what an agent \( a \) knows by what is true throughout a set \( R_a(w) \) of possibilities that are compatible with her knowledge in world \( w \):

\[
\mathcal{M}, w \models K_a \phi \iff \forall v \in R_a(w) : \mathcal{M}, v \models \phi. \tag{1}
\]

Q: what exactly does it mean for \( v \) to be “compatible”?

A: \( v \in R_a(w) \) means that everything \( a \) knows in \( w \) is true in \( v \).

Q: So you’re explaining when ‘\( K_a \phi \)’ is true in terms of knowledge itself?

A: Yes.

Q: Isn’t that a problem? Somehow cheating or conceptually circular?

A: No! The point of basic epistemic models is to represent the content of an agent’s knowledge—what the agent knows—not to give a substantive account of what it takes to know something in other terms.
Epistemology & What It Takes to Know

Notice that the way of modeling the content of an agent’s knowledge in basic epistemic-logical models gives no instructions re the question:

In general, if one wants to know that something is true—not just believe or wish it to be true—what does it take to achieve that?
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Notice that the way of modeling the *content* of an agent’s knowledge in basic epistemic-logical models gives no instructions re the question:

In general, if one wants to *know* that something is true—not just believe or wish it to be true—*what does it take to achieve that?*

Philosophers have long tried to provide some guidance here.
Epistemology & What It Takes to Know

Notice that the way of modeling the content of an agent’s knowledge in basic epistemic-logical models gives no instructions re the question:

In general, if one wants to know that something is true—not just believe or wish it to be true—what does it take to achieve that?

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Motivation for Today

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Many philosophers have realized this, e.g.:

- J.L. Austin. 1946. “Other Minds."

“If knowledge required the elimination of all logically possible alternatives, there would be no knowledge (at least of contingent truths).”

(Goldman 1976, 775)
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“There are always, it seems, possibilities that our evidence is powerless to eliminate... If knowledge...requires the elimination of all competing possibilities (possibilities that contrast with what is known), then, clearly we seldom, if ever, satisfy the conditions for applying the concept.”

(Dretske 1981, 365)
Relevant Alternatives and Subjunctivism

Austin, Goldman, Dretske, Lewis, and others propose a different idea:

According to relevant alternatives theories of knowledge, the agent should keep making observations of the world until she eliminates every relevant \( \neg p \)-possibility; an RA theorist then gives an account of what factors determine which possibilities are relevant and which are not.
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According to subjunctivist theories of knowledge, the agent should keep investigating the world until her belief about \( p \) becomes sufficiently correlated with the truth of \( p \) that, e.g., the following counterfactual is true: if \( p \) were false (in the “closest worlds”), she wouldn’t believe it.
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Claim: these ideas are not just the work of philosophers’ imaginations, detached from real life. To the contrary, arguably considerations of relevance and counterfactuals can help to explain why people are actually willing or unwilling to attribute knowledge to other people in practice.
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What do these theories have to do with epistemic logic? Answer: they bear on the correctness of core principles of epistemic logic.
Logical Omniscience vs. Epistemic Closure

Let’s distinguish two very different objections to

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(Aside: one way to bypass the logical omniscience problem is to read \( K\varphi \) as “\( \varphi \) is entailed by what the agent knows.” Then \( K \) is fine.)

Second, the denial of epistemic closure: even if agents did always put two and two together, \( K \) should still be rejected in its fullest generality for knowledge (though for circumscribed applications it might be fine).
According to Goldman, Dretske, and many others, the infallibilist view that for every proposition $p$, one must eliminate all $\neg p$-possibilities in order to know $p$ entails skepticism: we don’t know anything.
Fallibilism, Closure, and Contradiction

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Rejecting infallibilism, they hold the **fallibilist** view that for at least some $p$, there are some $\neg p$-possibilities that one does not have to eliminate in order to know $p$. 
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$$ (Kp \land K(p \rightarrow \neg s)) \rightarrow K\neg s, $$

which says that knowing $p$ \textit{does} require knowing $\neg s$, which requires eliminating $s$-possibilities (assuming that knowing something requires eliminating at least \textit{some} counter-possibilities).
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$$(Kp \land K(p \Rightarrow \neg s)) \Rightarrow K\neg s,$$

which says that knowing $p$ does require knowing $\neg s$, which requires eliminating $s$-possibilities (assuming that knowing something requires eliminating at least some counter-possibilities). **Contradiction?**
**Relevant Alternatives and Subjunctivism**

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But we can do something about this: add the extra structure to basic epistemic models and then investigate the theories mathematically.
A Family of Theories

I will begin by providing model-theoretic, logical formalizations of a family of relevant alternatives (RA) and subjunctivist theories of knowledge:

- **L** the RA theory of David Lewis (1996);
- **D** one way of interpreting the RA theory of Fred Dretske (1981);
- **H** the RA theory of Mark Heller (1989, 1999);
- **N** the basic tracking theory of Robert Nozick (1981);
- **S** and the safety theory of Ernest Sosa (1999).
Example (Medical Diagnosis)
Two medical students, A and B, are subjected to a test. Their professor introduces them to the same patient, who presents various symptoms, and the students are to make a diagnosis of the patient’s condition. After some independent investigation, both students conclude that the patient has a common condition $c$. In fact, they are both correct.
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The professor wished to see if the students would check for another common condition $c'$, which causes the same visible symptoms as $c$. While student A ran laboratory tests to rule out $c'$ before making the diagnosis of $c$, the student B made the diagnosis of $c$ after only a physical exam, having never considered the possibility of $c'$. 
RA-style analysis:

Although both students gave the correct diagnosis of $c$, student B did not *know* that the patient’s condition was $c$, since he did not rule out the alternative of $c'$. 
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- Student A secured against this possibility of error by running the laboratory tests.

Of course, student A did not secure against every possibility of error...
Example (Medical Diagnosis continued)

Suppose there is an extremely rare disease $x^1$ such that people with disease $x$ appear to have $c$ on many laboratory tests, even though people with $x$ are completely immune to $c$, and only extensive further testing can detect the presence or absence of $x$ in its early stages.

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Example (Medical Diagnosis continued)

Suppose there is an extremely rare disease $x^1$ such that people with disease $x$ appear to have $c$ on many laboratory tests, even though people with $x$ are completely *immune* to $c$, and only extensive further testing can detect the presence or absence of $x$ in its early stages.

Should we say that student A did not know that the patient’s condition was $c$ after all, since she did not rule out the possibility of $x$?

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RA-style analysis cont.:

- The requirement that one rule out all possibilities of error makes knowledge impossible, since there are always some possibilities of error—however remote and far-fetched—that are not eliminated by one’s evidence and experience.
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If the student has no reason to think that the patient may have the rare disease $x$, then it should not be necessary to rule out such a remote possibility in order to know that the patient has some common condition.
Suppose that all medical students know that people with $x$ have complete immunity to $c$, as assumed above, so

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Since the student did not run any tests that could possibly detect the presence or absence of \( x \), it would be unreasonable to claim that she knows that the patient does not have \( x \), so

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Then together with our judgment that the student knows that the patient has condition $c$,

$$K_A c,$$

we have a clear violation of closure under known implication,

$$(K_A c \land K_A (c \rightarrow \neg x)) \rightarrow K_A \neg x.$$
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- In order to know the premises $c$ and $c \rightarrow \neg x$, the agent must rule out certain relevant alternatives. In order to know the conclusion $\neg x$, the agent must also rule out certain relevant alternatives.

It is because the relevant alternatives may be different for the premises and the conclusion that closure under known implication does not hold in general.

Question: which other closure principles fail (or hold) on the RA theory?
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Validity Omniscience and Doxastic Closure

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- **Validity omniscience**: one knows all valid logical principles;

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One may satisfy the conditions for knowledge (ruling out the relevant alternatives, tracking the truth, etc.) with respect to some propositions and yet not with respect to all logical consequences of the set of those propositions, *even if* one has explicitly deduced all of these consequences.
A (single-agent) RA model is a tuple $\mathcal{M} = \langle W, \rightarrow, \preceq, V \rangle$, where:
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- $W = \{ w, v, u, \ldots \}$ is a non-empty set of “worlds” or “possibilities.”
A (single-agent) RA model is a tuple $\mathcal{M} = \langle W, \rightarrow, \preceq, V \rangle$, where:

- $V$: At $\rightarrow \mathcal{P}(W)$, where At $= \{p, q, r, \ldots\}$ is a fixed set of atomic sentence symbols, the same for all models.
A (single-agent) RA model is a tuple $\mathcal{M} = \langle W, \rightarrow, \preceq, V \rangle$, where:

- $\rightarrow$ is a reflexive relation on $W$ (note omitted loops in diagrams).
- $w \rightarrow v$ means possibility $v$ is **uneliminated** for the agent in $w$. 
A (single-agent) RA model is a tuple $\mathcal{M} = \langle W, \rightarrow, \preceq, V \rangle$, where:

- $\rightarrow$ is a reflexive relation on $W$ (note omitted loops in diagrams).
- Lewis (1996) proposes that “a possibility $\ldots \left[v\right] \ldots$ is uneliminated iff the subject’s perceptual experience and memory in $\ldots \left[v\right] \ldots$ exactly match his perceptual experience and memory in actuality” (553). Thus, for Lewis $\rightarrow$ is an equivalence relation.

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Whether we assume transitivity, symmetry, Euclideaness, etc., in addition to reflexivity will not affect our main results.
A (single-agent) RA model is a tuple $\mathcal{M} = \langle W, \rightarrow, \preceq, V \rangle$, where:

- $\preceq$ assigns to each $w \in W$ a binary relation $\preceq_w$ on some $W_w \subseteq W$:
  1. $\preceq_w$ is reflexive and transitive (preorder);
  2. $w \in W_w$, and for all $v \in W_w$, $w \preceq_w v$ (weak centering).

$u \preceq_w v$ means $u$ is at least as relevant (at $w$) as $v$ is;
A (single-agent) RA model is a tuple $\mathcal{M} = \langle W, \rightarrow, \leq, V \rangle$, where:

- $\leq$ assigns to each $w \in W$ a binary relation $\leq_w$ on some $W_w \subseteq W$;
  1. $\leq_w$ is reflexive and transitive (preorder);
  2. $w \in W_w$, and for all $v \in W_w$, $w \leq_w v$ (weak centering).

$u \leq_w v$ means $u$ is at least as relevant (at $w$) as $v$ is;
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$u \prec_w v$ iff $u \preceq_w v$ and not $v \preceq_w u$;

$u \equiv_w v$ iff $u \preceq_w v$ and $v \preceq_w u$. 
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Many epistemologists state conditions for knowledge in terms of some kind of (possibly context-dependent) ordering on worlds: e.g., Heller’s (1989) RA picture of “worlds surrounding the actual world ordered according to how realistic they are, so that those worlds that are more realistic are closer to the actual world than the less realistic ones” (25).

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The second condition corresponds to Lewis’s (1996) Rule of Actuality, that “actuality is always a relevant alternative” (554).
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  1. \( \preceq_w \) is reflexive and transitive (preorder);
  2. \( w \in W_w \), and for all \( v \in W_w \), \( w \preceq_w v \) (weak centering);
  3. if \( \emptyset \neq S \subseteq W_w \), then \( \text{Min}_{\preceq_w}(S) \neq \emptyset \) (well-founded).

For \( S \subseteq W \), \( \text{Min}_{\preceq_w}(S) = \left\{ v \in S \cap W_w \mid \neg \exists u \in S : u \prec_w v \right\} \).
Definition (RA Model)

A *relevant alternatives model* is a tuple \( M = \langle W, \rightarrow, \preceq, V \rangle \) where:

1. \( W \) is a non-empty set;
2. \( \rightarrow \) is a reflexive binary relation on \( W \);
3. \( \preceq \) assigns to each \( w \in W \) a binary relation \( \preceq_w \) on some \( W_w \subseteq W \);
   - 3.1 \( \preceq_w \) is reflexive and transitive (preorder);
   - 3.2 \( w \in W_w \), and for all \( v \in W_w \), \( w \preceq_w v \) (weak centering);
   - 3.3 if \( \emptyset \neq S \subseteq W_w \), then \( \text{Min}_{\preceq_w}(S) \neq \emptyset \) (well-founded);
4. \( V : \text{At} \rightarrow \mathcal{P}(W) \).
Definition (RA Model)

A *relevant alternatives model* is a tuple $\mathcal{M} = \langle \mathcal{W}, \rightarrow, \preceq, V \rangle$ where:

1. $\mathcal{W}$ is a non-empty set;
2. $\rightarrow$ is a reflexive binary relation on $\mathcal{W}$;
3. $\preceq$ assigns to each $w \in \mathcal{W}$ a binary relation $\preceq_w$ on some $\mathcal{W}_w \subseteq \mathcal{W}$;
   3.1 $\preceq_w$ is reflexive and transitive (preorder);
   3.2 $w \in \mathcal{W}_w$, and for all $v \in \mathcal{W}_w$, $w \preceq_w v$ (weak centering);
   3.3 if $\emptyset \neq S \subseteq \mathcal{W}_w$, then $\operatorname{Min}_{\preceq_w}(S) \neq \emptyset$ (well-founded);
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We will also consider *total* RA models in which for all $w \in \mathcal{W}$, $\preceq_w$ is a *total* preorder, i.e., $\forall u, v \in \mathcal{W}_w$, $u \preceq_w v$ or $v \preceq_w u$. 
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These models represent a fixed context. For context change, see:

*New Directions in Logic, Language, and Computation, LNCS.*
We could use our RA models to give semantics for a variety of different formal languages, but we’ll start with the basic modal language:

Definition (Epistemic Language)
The (single-agent) *epistemic language* $\mathcal{L}$ is defined inductively by

$$\varphi ::= p \mid \neg \varphi \mid (\varphi \land \varphi) \mid K\varphi,$$

where $p \in \text{At}$.

For simplicity in this talk, sometimes I will state results for the *flat* fragment in which only propositional formulas occur in the scope of $K$. 
Semantics

We will consider three semantics for the $K$ operator in RA models:
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- **C-semantics** for Cartesian. C-semantics is not intended to capture Descartes’ view of knowledge. Rather, it is supposed to reflect a high standard for the truth of knowledge claims—knowledge requires ruling out all possibilities of error, however remote—in the spirit of Descartes’ worries about error in the First Meditation.
Semantics

- **C-semantics** for Cartesian.
- **D-semantics** is one way (but not the only way or even the best way) of understanding Dretske’s (1981) RA theory, using Heller’s (1989, 1999) picture of relevance orderings over possibilities.
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- **L-semantics** is for Lewis’s (1999) RA theory (for a fixed context).
Semantics

Definition (Truth in an RA Model)

Given an RA model $\mathcal{M} = \langle W, \to, \preceq, V \rangle$, world $w \in W$, formula $\varphi$ in the epistemic language, and $x \in \{c, d, l\}$, we define $\mathcal{M}, w \models_x \varphi$ and $\llbracket \varphi \rrbracket_x^\mathcal{M} = \{ v \in W \mid \mathcal{M}, v \models_x \varphi \}$ as follows:

- $\mathcal{M}, w \models_x p$ iff $w \in V(p)$;
- $\mathcal{M}, w \models_x \neg \varphi$ iff $\mathcal{M}, w \not\models_x \varphi$;
- $\mathcal{M}, w \models_x (\varphi \land \psi)$ iff $\mathcal{M}, w \models_x \varphi$ and $\mathcal{M}, w \models_x \psi$.

- $\mathcal{M}, w \models_c K \varphi$ iff $\forall v \in \llbracket \varphi \rrbracket_c^\mathcal{M} : w \not\models v$;
- $\mathcal{M}, w \models_d K \varphi$ iff $\forall v \in \text{Min}_{\preceq} \llbracket \varphi \rrbracket_d^\mathcal{M} : w \not\models v$;
- $\mathcal{M}, w \models_l K \varphi$ iff $\forall v \in \text{Min}_{\preceq} (W) \cap \llbracket \varphi \rrbracket_l^\mathcal{M} : w \not\models v$,

where for $S \subseteq W$, $\overline{S} = \{ v \in W \mid v \not\in S \}$.

Wesley Holliday: Beyond Worlds and Accessibility
C-semantics and the Problem of Skepticism

\[ \mathcal{M}, w \models_c K \varphi \iff \forall v \in [\varphi]_c : w \not\models v. \]

\[ \mathcal{M}, w_1 \not\models_c Kc \]

C-semantics looks like the standard epistemic semantics in the tradition of Hintikka (1962); but now that we are explicitly representing relevance, it formalizes the infallibilist idea that all possibilities of error, however remote and far-fetched, must be eliminated—a view that leads to skepticism.
L-semantics and Non-skepticism

\[ \mathcal{M}, w \models K\varphi \text{ iff } \forall v \in \text{Min}_{\preceq_w}(W) \cap [\varphi]: w \not\models v. \]

L-semantics avoids skepticism by not requiring the elimination of every error possibility, but only those in the set \( \text{Min}_{\preceq_w}(W) \) of relevant worlds.
L-semantics and Non-skepticism

\[ \mathcal{M}, w \vDash K \varphi \text{ iff } \forall v \in \text{Min}_{\leq w}(W) \cap [\varphi]_I: w \not\vDash v. \]

\[ \mathcal{M}, w_1 \vDash K \neg x \]

L-semantics avoids skepticism by not requiring the elimination of every error possibility, but only those in the set Min_{\leq w}(W) of relevant worlds.
L-semantics and the Problem of Vacuous Knowledge

\[ M, w \models_I K \phi \text{ iff } \forall v \in \text{Min}_{\leq w}(W) \cap [\phi]_I: w \not\models v. \]

\[ M, w_1 \models_I K \neg x \]

With L-semantics, one can know \( \neg x \) even if \( x \) is possible and one has not eliminated any \( x \)-possibilities—even if one has not eliminated any possibilities at all. This would be knowledge of contingent empirical truths despite not having empirically eliminated any possibilities—vacuous knowledge.
D-semantics and Closure

\[ \mathcal{M}, w \models_d K \varphi \text{ iff for all } v \in \text{Min}_{\leq_w} (\llbracket \varphi \rrbracket_d), w \not\models v. \]

D-semantics avoids the skepticism of C-semantics, since knowing \( \varphi \) requires only that the closest \( \neg \varphi \)-worlds be eliminated, and avoids the vacuous knowledge of L-semantics, since if \( \neg \varphi \) is possible there will be some \( \neg \varphi \)-world(s) that must be eliminated. It also leads to denial of closure.
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\begin{center}
\begin{tikzpicture}
  \node[draw,shape=circle,fill=green!20] (c) at (0,0) {$c$};
  \node[draw,shape=circle,fill=green!20] (c') at (1,0) {$c'$};
  \node[draw,shape=circle,fill=green!20] (x) at (2,0) {$x$};
  \node[draw,shape=circle,fill=green!20] (c, x) at (3,0) {$c, x$};
  \node[draw,shape=circle,fill=gray] (w1) at (0,-1) {$w_1$};
  \node[draw,shape=circle,fill=gray] (w2) at (1,-1) {$w_2$};
  \node[draw,shape=circle,fill=gray] (w3) at (2,-1) {$w_3$};
  \node[draw,shape=circle,fill=gray] (w4) at (3,-1) {$w_4$};
  \draw[->] (c) to node[above] {$\approx_{w_1}$} (w1);
  \draw[->] (c') to node[above] {$\prec_{w_1}$} (w2);
  \draw[->] (x) to node[above] {$\prec_{w_1}$} (w3);
  \draw[->] (c, x) to node[above] {$\prec_{w_1}$} (w4);
\end{tikzpicture}
\end{center}

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\[ M, w \models_d K \phi \text{ iff } \forall v \in \text{Min}_{\leq w}(\left[ \phi \right]_d): w \nrightarrow v. \]

\[ M, w_1 \models_d Kc \quad M, w_1 \models_d K(c \rightarrow \neg x) \]

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Formalization: Relevant Alternatives

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\[ \mathcal{M}, w \models_d K \varphi \iff \forall v \in \text{Min}_{\leq w}(\llbracket \varphi \rrbracket_d): w \nrightarrow v. \]

\[ \mathcal{M}, w_{1} \models_d Kc \quad \mathcal{M}, w_{1} \models_d K(c \rightarrow \neg x) \quad \mathcal{M}, w_{1} \not\models_d K\neg x \]

**D-semantics avoids the skepticism of C-semantics**, since knowing \( \varphi \) requires only that the closest \( \neg \varphi \)-worlds be eliminated, and **avoids the vacuous knowledge of L-semantics**, since if \( \neg \varphi \) is possible there will be some \( \neg \varphi \)-world(s) that must be eliminated. It also leads to **denial of closure**.
D-semantics and Closure

\[ \mathcal{M}, w \models_d K\varphi \iff \forall v \in \text{Min}_{\leq w}(\overline{\varphi}_d): w \not\succ v. \]

\[ \mathcal{M}, s \not\models_d Kc \land K(c \rightarrow \neg x) \rightarrow K\neg x \]

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D-Semantics vs. L-Semantics

$$\mathcal{M}, w \models_d K \varphi \text{ iff } \forall v \in \text{Min}_{\leq w}(\overline{\varphi}_d): w \not\models v.$$  

$$\mathcal{M}, w \models_I K \varphi \text{ iff } \forall v \in \text{Min}_{\leq w}(W) \cap \overline{\varphi}_I: w \not\models v.$$  

**Fact (Known Implication)**

$$K \varphi \land K(\varphi \rightarrow \psi) \rightarrow K\psi \text{ is C/L-valid but not D-valid.}$$
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Yet Dretske wants knowledge to be semi-penetrating: “it seems to me fairly obvious that if someone knows that \( P \) and \( Q \), he thereby knows that \( Q \)” and “If he knows that \( P \) is the case, he knows that \( P \) or \( Q \) is the case” (1009). This is the “trivial side” of Dretske’s thesis.
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However, if we understand the RA theory according to D-semantics, based on Heller’s picture, then even these closure principles fail.
D-Semantics and the Problem of Containment

While some welcome the failure of closure under known implication, D-semantics faces a problem of containment: closure failures spread, and they spread to where no one wants them.
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\[\mathcal{M}, w \models_d K\varphi \text{ iff } \forall v \in \text{Min}^{\leq w}(\lceil \varphi \rceil_d): w \nvdash v.\]

\[\mathcal{M}, w \not\models_d K(c \land \neg x) \rightarrow K\neg x\]
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While some welcome the failure of closure under known implication, D-semantics faces a **problem of containment**: closure failures spread, and they spread to where no one wants them.

\[ \mathcal{M}, w \models_d K \varphi \text{ iff } \forall v \in \operatorname{Min}_{\preceq_w}([\varphi]_d) : w \not \not \! v. \]

\[ \mathcal{M}, w_1 \not \not \!_d K(c \land \neg x) \rightarrow K\neg x \]

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\[ M, w_1 \models_D Kc \]
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D-Semantics and the Problem of Containment

\[ M, w \models_d K \varphi \text{ iff } \forall v \in \text{Min}_{\leq_w} ([\varphi]_d): w \not\sim v. \]

Fact (Distribution and Addition)

\( K(\varphi \land \psi) \rightarrow K\varphi \) and \( K\varphi \rightarrow K(\varphi \lor \psi) \) are C/L-valid, but not D-valid.
A Dretskean Dilemma

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- If we understand RA theory according to D-semantics, then \( K \) is not semi-penetrating, contrary to the “trivial side” of Dretske’s thesis.
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- If we understand the theory according to L-semantics, then \( K \) is fully-penetrating, contrary to the non-trivial side of Dretske’s thesis; and we have the problem of vacuous knowledge.

It is difficult to escape this dilemma while retaining the picture “of worlds arranged around the actual world in order of similarity, with those that are too far away from the actual world being irrelevant” (Heller 1999, 119).