1 How much do you know?

1. I have constructed a world in Tarski’s World using objects named a through f, but I’m not going to show it to you. Now consider the sentences below, and decide whether you can determine their truth value even though you can’t see my world. You should be able to explain why or why not.

(a) \(d=d\)
   
   We know that we can use the =Intro to introduce an identity at any point. Of course there must exist such an object in the world, but I have told you that objects a through f exist. So \(d=d\) must be true.

(b) \(\text{BackOf}(b,a) \lor \text{BackOf}(a,b)\)
   
   You cannot determine the truth value of this sentence without looking at the world. After all, if \(a\) and \(b\) are in the same row, then neither \(\text{BackOf}(b,a)\) nor \(\text{BackOf}(a,b)\) will be true.

(c) \(\text{LeftOf}(e,f) \land \text{RightOf}(f,e)\)
   
   \(\text{LeftOf}(e,f)\) and \(\text{RightOf}(f,e)\) are equivalent, and if \(e\) and \(f\) are in the same column, then these will not be true.

(d) \(b=a \lor a \neq b\)
   
   \(b=a\) is the same as \(a=b\), and \(a \neq b\) is its negation, so this sentence says that either \(a\) and \(b\) are identical objects, or they are not. One of the must be true, so the whole sentence is true, regardless of the world.

(e) \(\text{Large}(a) \lor \neg \text{Large}(a)\)
   
   Either \(a\) is large, or it is not. So this sentence must be true, regardless of the world.

2. (Based on Ex 3.14) Is \(\neg (\text{Small}(a) \lor \text{Small}(b))\) a logical consequence of \(\neg \text{Small}(a) \lor \text{Small}(b)\)? If it is, write an informal proof. If it is not, describe a counterexample.

   The following counterexample shows that \(\neg (\text{Small}(a) \lor \text{Small}(b))\) is not a logical consequence of \(\neg \text{Small}(a) \lor \text{Small}(b)\).

   Let \(a\) be a large object, and \(b\) be a small object. Then \(a\) is not small, so \(\neg \text{Small}(a)\) is true and therefore \(\neg \text{Small}(a) \lor \text{Small}(b)\) is true.
However, ¬(Small(a) ∨ Small(b)) is false, since (Small(a) ∨ Small(b)) is true, since Small(b) is true.

3. Is ¬(Small(a) ∨ Small(b)) a logical consequence of ¬Small(a) ∧ ¬Small(b)? If it is, write an informal proof. If it is not, describe a counterexample.

Yes. Suppose ¬Small(a) ∧ ¬Small(b) is true. Then both conjuncts must be true, that is ¬Small(a) is true and ¬Small(b) is true. But then Small(a) is false and so is Small(b). This just means that a is either medium or large, and b is either medium or large.

Since neither a nor b is small, Small(a) ∨ Small(b) is false, since neither disjunct is true. But then ¬Small(a) ∨ Small(b) must be true.

This shows that ¬Small(a) ∨ Small(b) is a logical consequence of ¬Small(a) ∧ ¬Small(b), since the former must be true whenever the latter is true.

4. Is ¬(Small(a) ∧ Small(b)) a logical consequence of ¬Small(a) ∨ ¬Small(b)? If it is, write an informal proof. If it is not, describe a counterexample.

Yes. Suppose ¬Small(a) ∨ ¬Small(b) is true. Then one of its disjuncts must be true, that is, either ¬Small(a) is true or ¬Small(b) is true. This means that either Small(a) is false or Small(b) is false. This is just to say that either a or b is not small.

But then it is obviously not the case that both a and b are small, so Small(a) ∧ Small(b) is false. This means that ¬(Small(a) ∧ Small(b)) is true, which is the conclusion we sought.

2 Something slightly harder, if there’s time.

(Based on Ex 3.13) For the following exercise you can either draw an 8x8 checkerboard on paper and indicate your objects on it, or work on it at home using Tarski’s World. (It will be more challenging to do it on paper.) The following sentences are taken from the file Schroder’s Sentences.sen. Your goal is to construct an exercise in which every sentence is true:

1. Dodec(a) ∧ Cube(b)
2. FrontOf(b,a) ∧ LeftOf(d,b) ∧ BackOf(f,d) ∧ RightOf(a,f)
3. Tet(c) ∧ Tet(e) ∧ LeftOf(c,d)
4. ¬LeftOf(c,e) ∧ ¬LeftOf(e,c)
5. Between(d,b,c) ∧ ¬Between(d,c,a)
6. ¬Cube(b) ∨ (Cube(d) ∧ BackOf(d,b))
7. \( \text{Small}(c) \land \text{FrontOf}(c, b) \lor \text{Cube}(d) \)

8. \( \text{Larger}(f, a) \land \text{Larger}(a, b) \)

9. \( \neg(\text{Smaller}(c, a) \lor \text{Smaller}(a, c)) \)

10. \( \text{Larger}(d, b) \land \text{Larger}(f, d) \land \neg(\text{Larger}(e, d) \lor \text{Larger}(d, e)) \)

The following world will work:

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3 Challenge questions: good luck with these!

The truth-functional sentential connectives that we learn about, such as \( \land \) and \( \neg \), all have counterparts in English. However, it is possible to construct a great many other connectives. Recall what it is for a sentential connective to be truth-functional. A binary sentential connective takes two sentences and outputs one truth value. A ternary sentential connective takes three sentences and outputs one truth value...and so on.

1. How many different binary sentential connectives could there possibly be? List their truth tables. How many of them have English counterparts?

Consider first how many lines a truth table should have for a binary sentential connective. The truth value of the whole thing depends on the truth value of each of the two component sentences. There are two truth values the first sentence can take, and for each of these there are two truth values the second sentence can take. (Look at a truth table and see how the truth values are arranged to reflect this.) So there are \( 2 \times 2 = 4 \) ways to assign truth values to the component sentences, and so every binary connective has a truth table with 4 lines.

But now consider how many ways we could draw up a truth table. For each of the four lines, we either write the output as True or False. So there are \( 2 \times 2 \times 2 = 2^4 = 16 \) ways to draw up a truth table. So there are 16 possible binary connectives, some of which you have seen, and some of which have no common English equivalent.

If you want to see all of these connective, look at Enderton’s *Mathematical Logic* 2nd edition.
2. **How many different ternary sentential connectives are there? How did you arrive at this number? You should not try to list them all!**

We calculate the number of ternary connectives in the same way as we calculated the number of binary connectives in the last question. A truth table for a ternary connective has $2 \times 2 \times 2 = 2^3 = 8$ lines. But this means that there are $2^8 = 256$ possible ways to draw up such a truth table, and so there are 256 different ternary connectives. Good luck coming up with English equivalents!

3. **The set of sentential connectives $\{\neg, \land, \lor\}$ is complete.** This means that every possible sentence using binary sentential connectives can be translated into a sentence that uses just these three connectives and has exactly the same truth conditions. The point is the following: suppose we had a language using all of the binary connectives that you found in question 1. We could take any sentence in this language and translate it into a more efficient language that only uses the three connectives $\neg$, $\land$, and $\lor$.

Can you show that the set $\{\neg, \lor\}$ is complete? In order to do this, show how an arbitrary sentence using the connectives $\{\neg, \land, \lor\}$ can be expressed using just the connectives $\neg$ and $\lor$. Call $\mathcal{L}_{old}$ the old language that contains the connectives $\neg$, $\land$ and $\lor$ and $\mathcal{L}_{new}$ the new language containing only the connectives $\neg$ and $\lor$. Now complete the following induction in order to show that every formula in $\mathcal{L}_{old}$ has an equivalent formula in $\mathcal{L}_{new}$

- If $\neg \phi$ is a formula in $\mathcal{L}_{old}$, then $\neg \phi$ is an equivalent formula in $\mathcal{L}_{new}$.
- If $\phi \lor \psi$ is a formula in $\mathcal{L}_{old}$, then $\phi \lor \psi$ is an equivalent formula in $\mathcal{L}_{new}$.
- If $\phi \land \psi$ is a formula in $\mathcal{L}_{old}$, then $\neg(\neg \phi \lor \neg \psi)$ is an equivalent formula in $\mathcal{L}_{new}$.

You can do it in the following way:

- If $\neg \phi$ is a formula in $\mathcal{L}_{old}$, then $\neg \phi$ is an equivalent formula in $\mathcal{L}_{new}$.
- If $\phi \lor \psi$ is a formula in $\mathcal{L}_{old}$, then $\phi \lor \psi$ is an equivalent formula in $\mathcal{L}_{new}$.
- If $\phi \land \psi$ is a formula in $\mathcal{L}_{old}$, then $\neg(\neg \phi \lor \neg \psi)$ is an equivalent formula in $\mathcal{L}_{new}$.

Similarly, it is possible to show that the set $\{\neg, \land\}$ is complete.

4. **Now, it is possible to show that there is a set containing just one connective that is complete. Can you find this set? (Note: it doesn’t contain any of the traditional connectives you have learnt about.)**

The connective $|$ is sometimes called the ‘Sheffer stroke’, and should be interpreted so that $p|q$ translates as ‘not both $p$ and $q’$. Here is the truth table:

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And here is how we can inductively define a translation from the language $L_{\text{old}}$ containing connectives $\{\neg, \land, \lor\}$ to a language $L_{\text{new}}$ containing just $\{\}\$:

- If $\neg \phi$ is a formula in $L_{\text{old}}$, then $\phi | \phi$ is an equivalent formula in $L_{\text{new}}$.
- If $\phi \lor \psi$ is a formula in $L_{\text{old}}$, then $(\phi | \phi) | (\psi | \psi)$ is an equivalent formula in $L_{\text{new}}$.
- If $\phi \land \psi$ is a formula in $L_{\text{old}}$, then $(\phi | \psi) | (\phi | \psi)$ an equivalent formula in $L_{\text{new}}$.

5. We have been describing the sentential connectives in terms of their truth values True and False. Some logic texts replace False with 0 and True with 1. If we do this, we can describe the sentential connectives algebraically. So, for example, for any two sentences $\phi$ and $\psi$ having truth values $[\phi]$ and $[\psi]$ we could describe a connective that produces as output the value $[\phi] + [\psi]$. If the sentences were both true, this would produce the output $[\phi] + [\psi] = 1 + 1 = 0$. (We need to do the math in binary.) Now determine algebraic representations for $\neg$, $\lor$ and $\land$.

- $[\neg \phi] = 1 + [\phi]$
- $[\phi \lor \psi] = \max([\phi], [\psi])$
- $[\phi \land \psi] = [\phi] \times [\psi]$

When you learn the truth table for $\rightarrow$, you should ask yourself whether you can represent it algebraically.